Lecture 8

SWAP test on multiple qubits & reducing circuit depth in the algorithm for increased circuit width.

suppose we have qubits in systems $A_1, \ldots, A_n \equiv A$ & in systems $B_1, \ldots, B_n \equiv B$

If we would like to swap A & B, we can swap individually

That is,

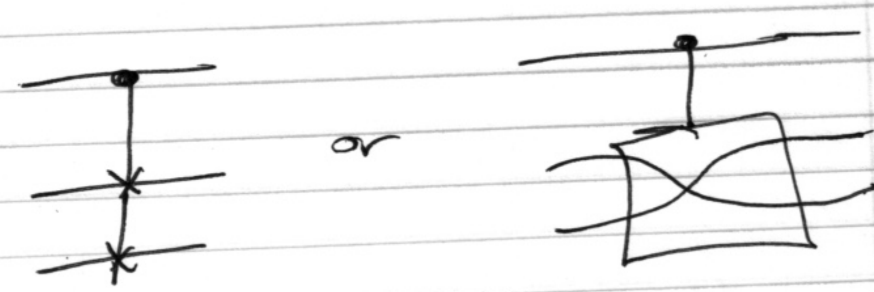
$$SWAP_{AB} = SWAP_{A_1B_1} \otimes SWAP_{A_2B_2} \otimes \cdots \otimes SWAP_{A_nB_n}$$

Now suppose that we have access to controlled-SWAP gate as a basic primitive (acting only on qubits)

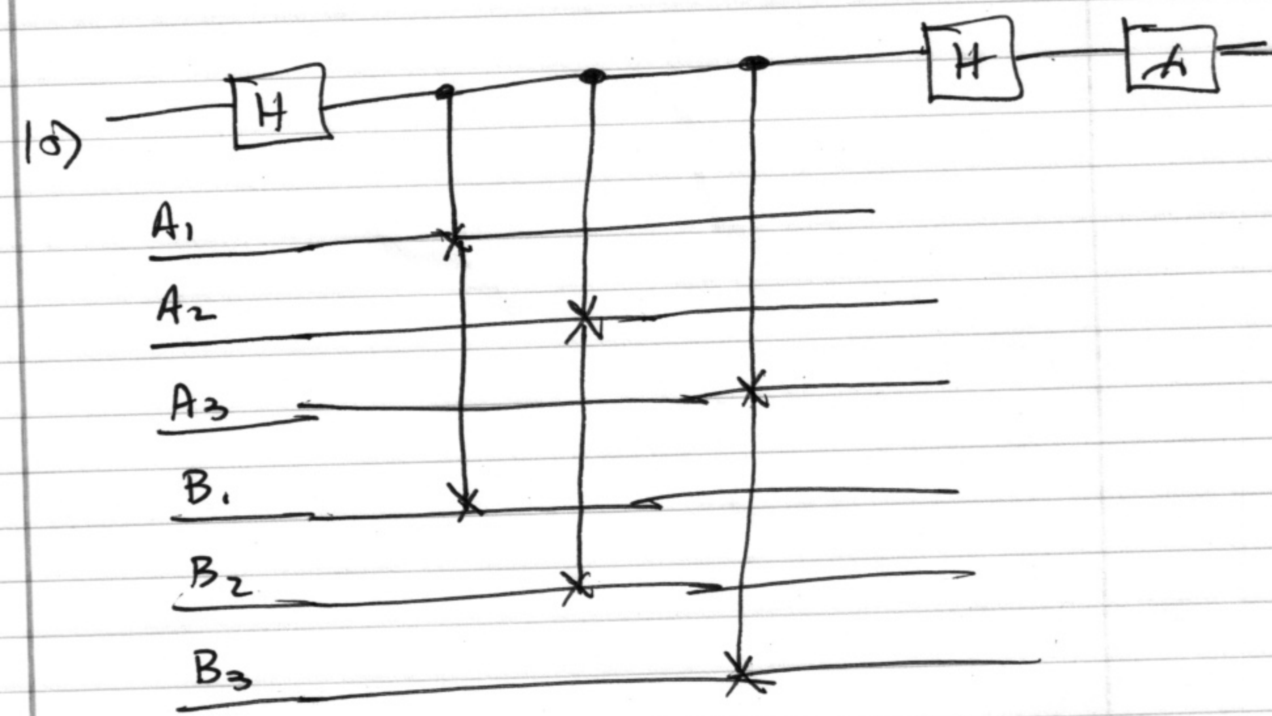
This unitary is as follows:

$$|0\rangle\langle 0|_C \otimes I_{A_1 B_1} + |1\rangle\langle 1|_C \otimes SWAP_{A_1 B_1}$$

& written in circuit notation as



then suppose we would like to implement the SWAP test for 3 qubits each. Then we can do it as follows:



By similar reasoning as before,

$$Pr\{0\} = \frac{1 + |\langle\psi|\varphi\rangle|^2}{2}$$

where $|\psi\rangle_{A_1A_2A_3}$ &

$|\varphi\rangle_{B_1B_2B_3}$

& $$Pr\{1\} = \frac{1 - |\langle\psi|\varphi\rangle|^2}{2}$$

Here we see that the circuit depth is linear in the # of qubits. Is there a way to reduce the circuit depth?

Consider that the bottleneck is that the controls of the SWAP gates all act on the same control qubit.

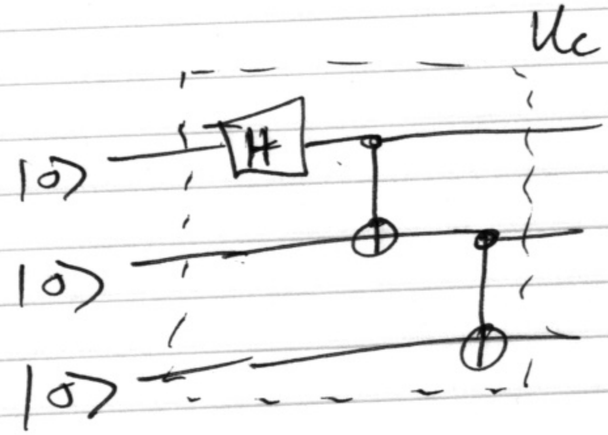
We can make this part of the circuit have <u>constant</u> depth by a technique we can call q. fan-out

Consider instead preparing
a GHZ state of three
qubits & having three
control qubits instead of
a single one.

circuit to encode a GHZ state

$$\frac{|000\rangle + |111\rangle}{\sqrt{2}}$$

B



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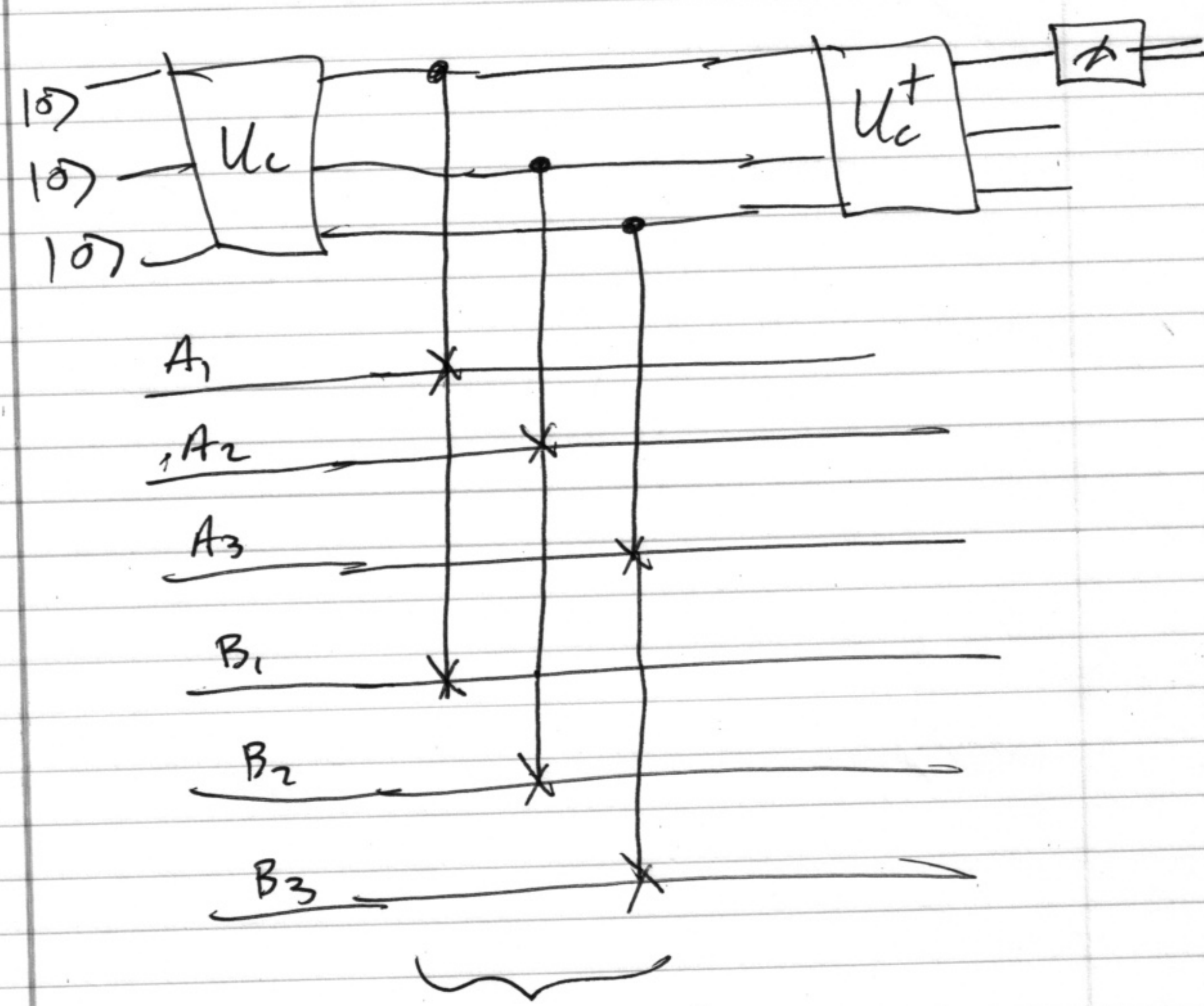
Then we can modify the circuit to be
SWAP test



these gates can be run in parallel & so this part now has constant depth.

this implies that this part of the algorithm will be faster & give fewer errors

Let us now analyze the algorithm

Initial state is

$$|000\rangle |\psi\rangle_{A_1A_2A_3} |\varphi\rangle_{B_1B_2B_3}$$

After U_c

$$\rightarrow \frac{|000\rangle + |111\rangle}{\sqrt{2}} |\psi\rangle_A |\varphi\rangle_B$$

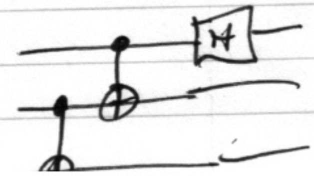
$$= \cancel{\frac{1}{\sqrt{2}} |000\rangle +}$$

$$\frac{1}{\sqrt{2}}\left(|000\rangle|\psi\rangle_A|\varphi\rangle_B + |111\rangle|\psi\rangle_A|\varphi\rangle_B\right)$$

After controlled - SWAPs,
keeping in mind the local actions,

$$\rightarrow \frac{1}{\sqrt{2}}\left(|000\rangle|\psi\rangle_A|\varphi\rangle_B + |111\rangle|\varphi\rangle_A|\psi\rangle_B\right)$$

Now apply $U_c^\dagger$ to control qubits,
keeping in mind that

$U_c^\dagger =$ 

After the CNOTs, the state becomes

$$\rightarrow \frac{1}{\sqrt{2}}\left(|000\rangle|\psi\rangle_A|\varphi\rangle_B + |100\rangle|\varphi\rangle_A|\psi\rangle_B\right)$$

$$= \frac{1}{\sqrt{2}}\left(|0\rangle|\psi\rangle_A|\varphi\rangle_B + |1\rangle|\varphi\rangle|\psi\rangle_B\right)|00\rangle$$

these qubits get factored out

Then rest of analysis proceeds as before & we conclude

that $\Pr[0] = \dfrac{1 + |\langle\psi|\varphi\rangle|^2}{2}$

$\Pr[1] = \dfrac{1 - |\langle\psi|\varphi\rangle|^2}{2}$

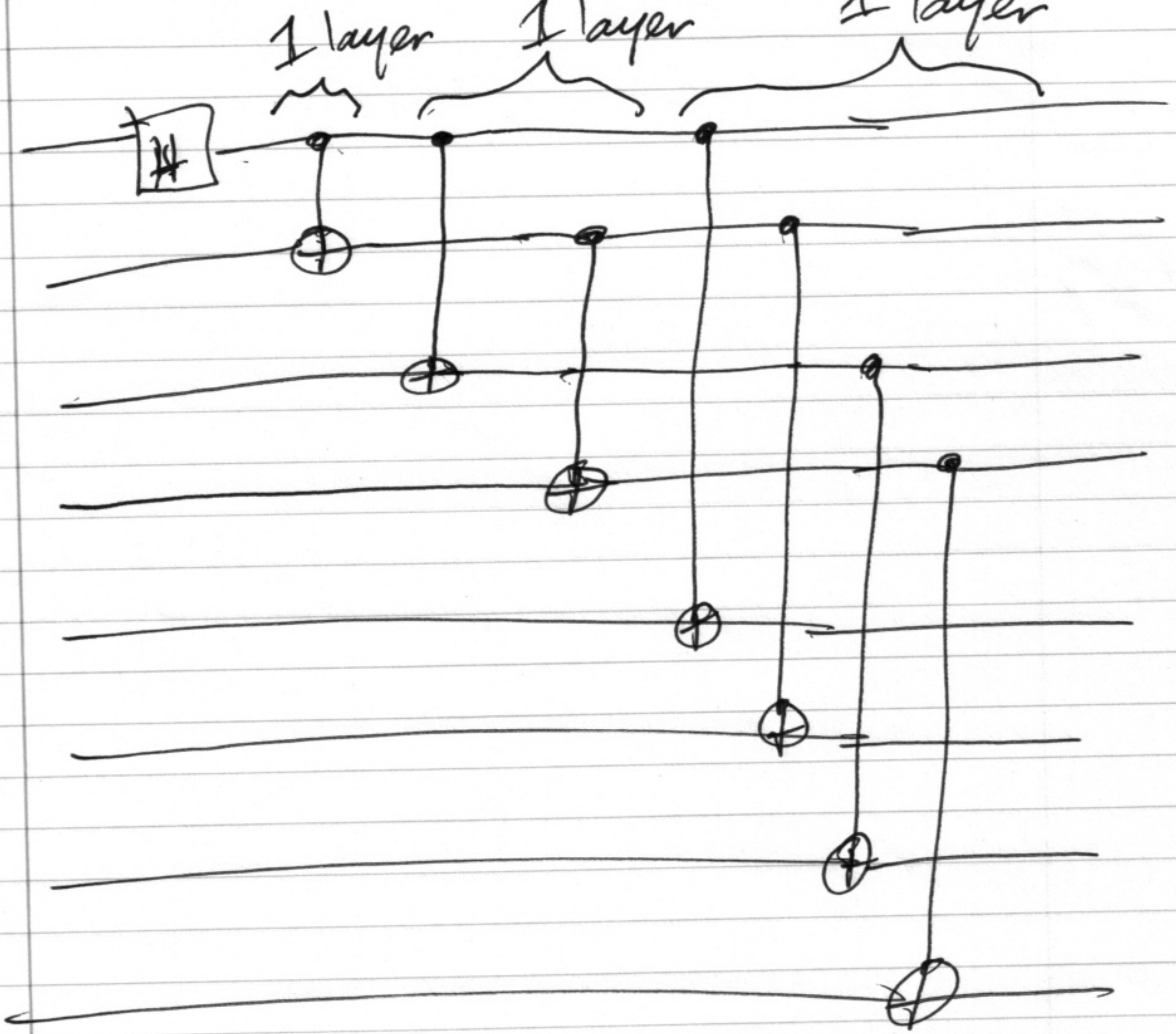
the circuit depth, as of now,

~~there is a~~ for preparing

the GHZ state, is

logarithmic depth.

General construction is

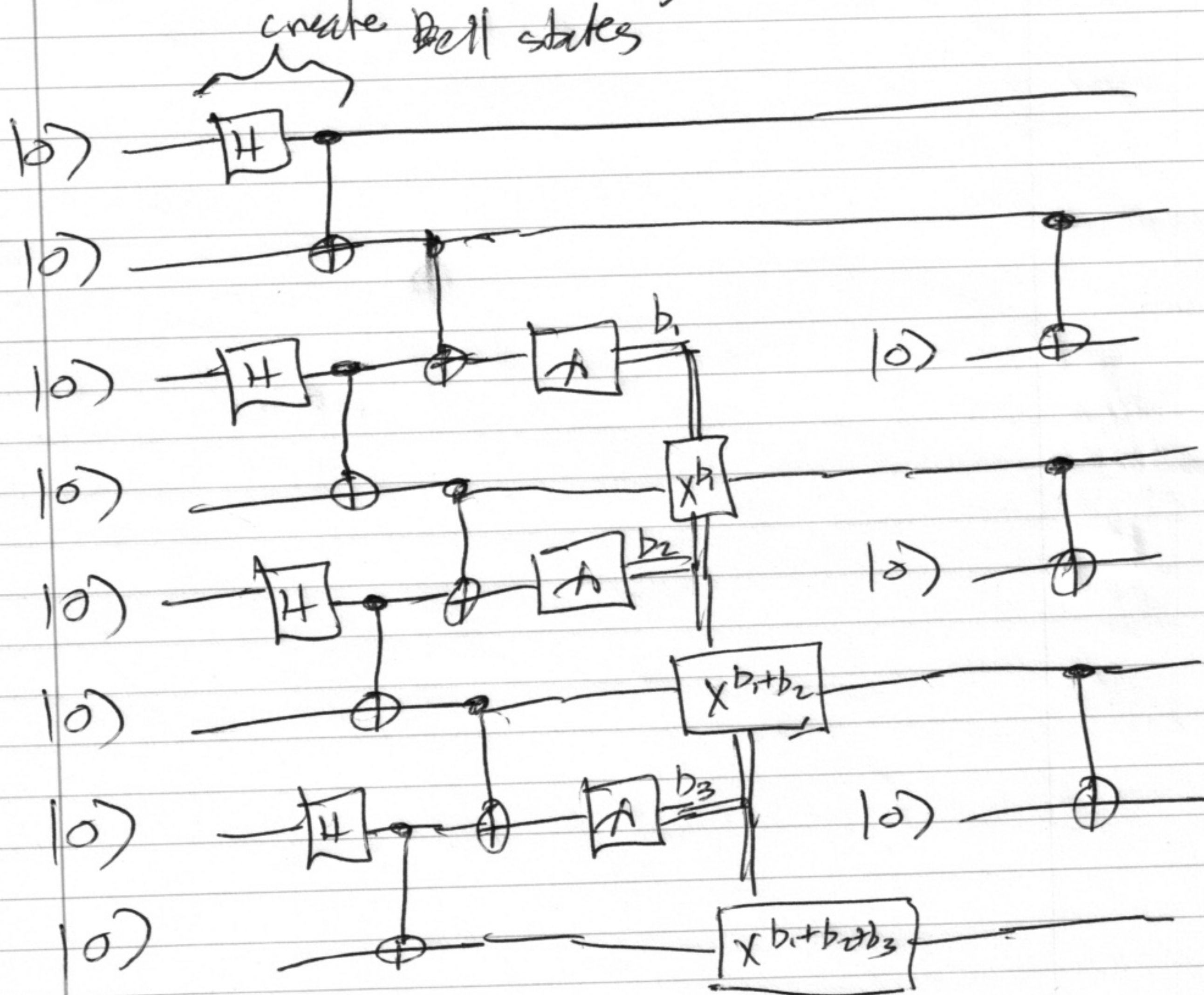


Can we do better?

Yes!

We can prepare GHZ states in constant quantum depth w/ mid-circuit measurements

Consider the following circuit



this is a constant quantum depth circuit

Why does this work?

Consider a simple example

$$|0\rangle|0\rangle|0\rangle|0\rangle$$

$$\to |\Phi^+\rangle|\Phi^+\rangle$$

$$= \frac{1}{\sqrt{2}}\big(|00\rangle + |11\rangle\big) \otimes \frac{1}{\sqrt{2}}\big(|00\rangle + |11\rangle\big)$$

$$= \frac{1}{2}\big(|0000\rangle + |1100\rangle + |0011\rangle + |1111\rangle\big)$$

after CNOT

$$\to \frac{1}{2}\big(|0000\rangle + |1110\rangle + |0011\rangle + |1101\rangle\big)$$

measuring the third qubit

gives

$$\frac{1}{\sqrt{2}}\big(|000\rangle + |111\rangle\big) \text{ w/ prob. } 1/2$$

$$\frac{1}{\sqrt{2}}\left(|110\rangle + |001\rangle\right) \quad \text{w/ prob. } 1/2$$

if second happens, then apply

X to third qubit

to get

$$\frac{1}{\sqrt{2}}\left(|000\rangle + |111\rangle\right)$$

circuit provided generalizes this construction

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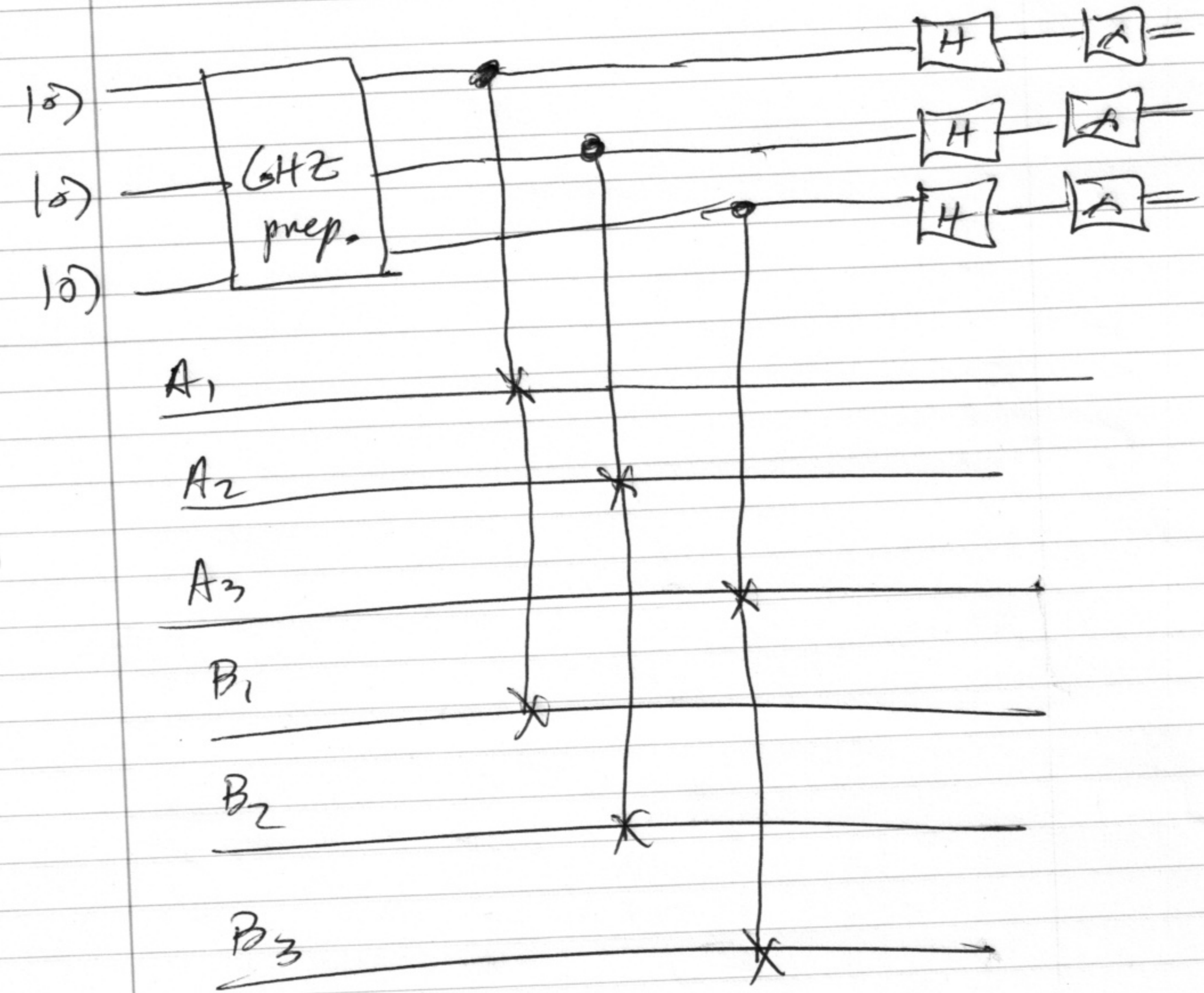
so now the preparation is constant depth and so are controlled SWAPs.

What about final part?

Modify to be as follows:



As before, after controlled SWAPs, state is

$$\frac{1}{\sqrt{2}}\left(|000\rangle|\psi\rangle|\varphi\rangle + |111\rangle|\varphi\rangle|\psi\rangle\right)$$

what happens after measuring in Hadamard basis?

Define $|\tilde{x}\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + (-1)^x|1\rangle\right)$

for $x \in \{0,1\}$

probability of getting outcome $x_1 x_2 x_3$ is

$$\left\| \langle \tilde{x}_1 \tilde{x}_2 \tilde{x}_3 | \frac{1}{\sqrt{2}}\left(|000\rangle|\psi\rangle|\varphi\rangle + |111\rangle|\varphi\rangle|\psi\rangle\right)\right\|_2^2$$

$$= \left\| \frac{1}{\sqrt{2^4}} \left(|\psi\rangle|\varphi\rangle + (-1)^{x_1+x_2+x_3} |\varphi\rangle|\psi\rangle \right) \right\|_2^2$$

$$= \frac{1}{2^4} \left(2 + (-1)^{x_1+x_2+x_3} \, 2 \, |\langle\psi|\varphi\rangle|^2 \right)$$

$$= \frac{1}{2^3} \left(1 + (-1)^{x_1+x_2+x_3} |\langle\psi|\varphi\rangle|^2 \right)$$

Now set a R.V.

$$Z = (-1)^{x_1+x_2+x_3}$$

after observing measurement outcomes x_1, x_2, x_3.

Then

$$\mathbb{E}[Z] = \sum_{x_1, x_2, x_3} \Pr[x_1, x_2, x_3] \, (-1)^{x_1+x_2+x_3}$$

$$= \sum_{x_1, x_2, x_3} \frac{1}{2^3} \left(1 + (-1)^{x_1+x_2+x_3} |\langle\psi|\varphi\rangle|^2 \right) \cdot (-1)^{x_1+x_2+x_3}$$

$$= \frac{1}{2^3} \sum_{x_1, x_2, x_3} \left[(-1)^{x_1+x_2+x_3} + |\langle\psi|\varphi\rangle|^2 \right]$$

$$= |\langle\psi|\varphi\rangle|^2 \quad \Rightarrow \mathbb{E}[z] = |\langle\psi|\varphi\rangle|^2$$

So this gives a constant depth circuit to evaluate $|\langle\psi|\varphi\rangle|^2$