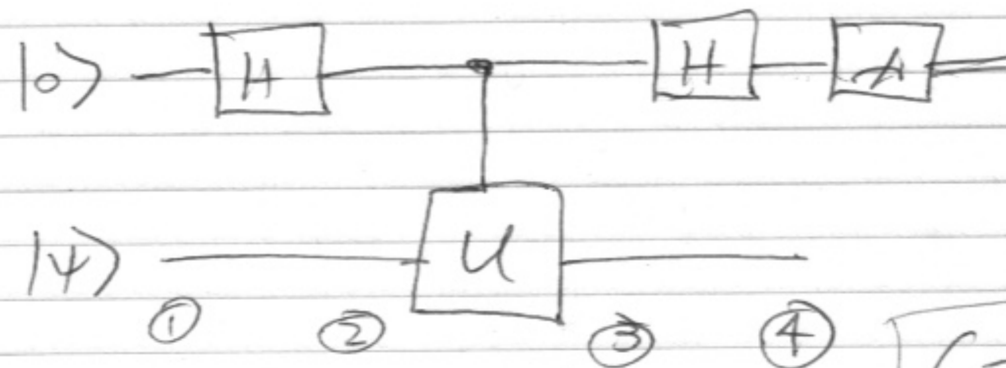
Lecture 7

We can apply reasoning from before to a generic controlled-unitary.

Consider the following circuit



$C\text{-}U = |0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes U$

Analyze again step by step

① $|0\rangle|\psi\rangle$

→ ② $|+\rangle|\psi\rangle = \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right)|\psi\rangle = \frac{|0\rangle|\psi\rangle + |1\rangle|\psi\rangle}{\sqrt{2}}$

→ ③ $\frac{|0\rangle|\psi\rangle + |1\rangle U|\psi\rangle}{\sqrt{2}}$

2

(4) $\dfrac{|+\rangle|\psi\rangle + |-\rangle U|\psi\rangle}{\sqrt{2}}$

probability to get outcome zero:

$$p(0) = \left\| \left(|0\rangle\langle 0| \otimes I\right) \frac{1}{\sqrt{2}} \left(|+\rangle|\psi\rangle + |-\rangle U|\psi\rangle \right) \right\|_2^2$$

$$= \left\| |0\rangle \left(\frac{|\psi\rangle + U|\psi\rangle}{2} \right) \right\|_2^2$$

$$= \left\| |0\rangle \left(\frac{I+U}{2} \right) |\psi\rangle \right\|_2^2$$

$$= \left[\langle 0| \langle\psi| \left(\frac{I+U^\dagger}{2} \right) \right] \left[|0\rangle \left(\frac{I+U}{2} \right) |\psi\rangle \right]$$

$$= \frac{1}{4} \langle\psi| (I+U^\dagger)(I+U) |\psi\rangle$$

$$= \frac{1}{4} \langle\psi| 2I + U + U^\dagger |\psi\rangle$$

$$= \frac{1}{4} \left(2 + \langle\psi|U|\psi\rangle + \langle\psi|U^\dagger|\psi\rangle \right)$$

$$= \frac{1}{2} \left(1 + \mathrm{Re}\left[\langle\psi|U|\psi\rangle \right] \right)$$

Check: Is this a legitimate probability?

Consider that

$$\mathrm{RE}[\langle\psi|U|\psi\rangle] \le |\langle\psi|U|\psi\rangle|$$

$$\le \| |\psi\rangle \|_2 \, \| U|\psi\rangle \|_2$$

$$= 1$$

$$\cancel{\Rightarrow \tfrac{1}{2}\left(1+\mathrm{RE}[\langle\psi|U|\psi\rangle]\right)}$$

Same reasoning gives

$$-\mathrm{RE}\{\langle\psi|U|\psi\rangle\} \le 1$$

$$\Rightarrow \mathrm{RE}\{\langle\psi|U|\psi\rangle\} \ge -1$$

$$\Rightarrow \frac{1}{2}\left(1+\mathrm{RE}[\langle\psi|U|\psi\rangle]\right) \in [0,1]$$

✓

What is the probability of getting outcome 1?

$$p(1) = \left\| \left(|1\rangle\langle 1| \otimes I \right) \frac{1}{\sqrt{2}} \left(|+\rangle|\psi\rangle + |-\rangle U|\psi\rangle \right) \right\|_2^2$$

$$= \left\| |1\rangle \left(\frac{I - U}{2} \right) |\psi\rangle \right\|_2^2$$

$$= \frac{1}{2} \left(1 - \mathrm{RE}\left[\langle\psi| U |\psi\rangle \right] \right)$$

<u>Question:</u> Describe a q. algorithm for estimating $\mathrm{RE}\left[\langle\psi|U|\psi\rangle\right]$ w/in additive error ε & w/ success probability at least $1-\delta$.

Now let us consider a special input state & a special choice of unitary

pick input state to be

$$|\psi\rangle = |\varphi\rangle \otimes |\phi\rangle \quad \&$$

unitary to be $U = \mathrm{SWAP}$, which acts as

$$\mathrm{SWAP}\ |\varphi\rangle \otimes |\phi\rangle = |\phi\rangle \otimes |\varphi\rangle$$

(see homework)

then

$$\langle\psi|U|\psi\rangle$$
$$= \langle\varphi| \otimes \langle\phi|\ \mathrm{SWAP}\ |\varphi\rangle \otimes |\phi\rangle$$
$$= \big(\langle\varphi| \otimes \langle\phi|\big)\big(|\phi\rangle \otimes |\varphi\rangle\big)$$
$$= \langle\varphi|\phi\rangle\langle\phi|\varphi\rangle = |\langle\varphi|\phi\rangle|^2$$

This is called the SWAP test (6)

Implies that this algorithm can estimate the inner product of $|\psi\rangle$ & $|\phi\rangle$.

One of the most important tasks in q. machine learning.

Idea is to encode classical information into high-dimensional states & then use this basic algorithm to estimate the inner product $|\langle\psi|\phi\rangle|^2$.
If the states are complex (i.e., difficult to simulate on a classical computer), then the idea is that this could present

an advantage for QML.

In particular, the method of support vector machines evaluates inner products of high-dimensional vectors as a basic subroutine.

There is another way to estimate inner products & it is called the destructive SWAP test.

Let us consider the case of two qubits $|\psi\rangle$ & $|\phi\rangle$

consider from homework 2 that

$$\mathrm{Tr}\{\mathrm{SWAP}\, |\varphi\rangle\langle\varphi| \otimes |\phi\rangle\langle\phi|\}$$

$$= \mathrm{Tr}\{|\varphi\rangle\langle\varphi|\phi\rangle\langle\phi|\}$$

$$= \langle\phi|\varphi\rangle\langle\varphi|\phi\rangle$$

$$= |\langle\phi|\varphi\rangle|^2$$

we can write first line as

$$\langle\psi| \,\mathrm{SWAP}\, |\psi\rangle$$

where $|\psi\rangle = |\varphi\rangle \otimes |\phi\rangle$

this is an expectation of the SWAP observable.

From homework, we know that for qubits,

$$\mathrm{SWAP} = \Phi^+ + \Phi^- + \Psi^+ - \Psi^-$$

where we use the shorthand $\Phi^+ \equiv |\Phi^+\rangle\langle\Phi^+|$, etc.

$\Rightarrow$ $\langle\psi|\,\text{SWAP}\,|\psi\rangle$

$$= \langle\psi|\Phi^+|\psi\rangle + \langle\psi|\Phi^-|\psi\rangle + \langle\psi|\Psi^+|\psi\rangle - \langle\psi|\Psi^-|\psi\rangle$$

$$= |\langle\Phi^+|\psi\rangle|^2 + |\langle\Phi^-|\psi\rangle|^2 + |\langle\Psi^+|\psi\rangle|^2 - |\langle\Psi^-|\psi\rangle|^2$$

Given this, what is a q. algorithm to estimate $\langle\psi|\,\text{SWAP}\,|\psi\rangle$?

(back to first weeks of class)

Set $i=1$.

Answer: Perform Bell measurement, set $z_i = 1$ if outcome Φ^+, Φ^-, Ψ^+ & $z_i = -1$ if outcome Ψ^-

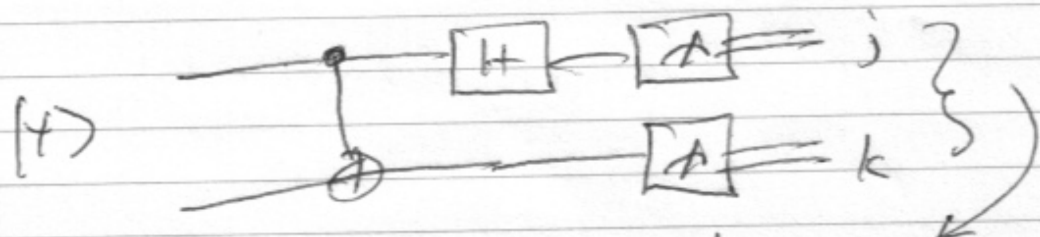
Increment i.

repeat for sufficiently many times as required by Hoeffding bound.

Set $\bar{Z}^n = \frac{1}{n}\sum_{i=1}^{n} Z_i$

as estimate of $\langle\psi|SWAP|\psi\rangle$

Circuit looks like this:



outcomes of Bell measurement

called destructive SWAP test because state is destroyed in the process.

Multi-qubit generalization available on page 9 of arXiv:2309.02515