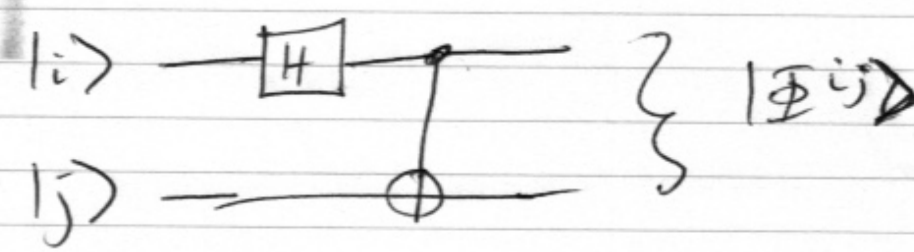
# Lecture 6

Basic protocol: superdense coding

can view as a basic q. computation
or a basic communication protocol

- Last we talked about a q. circuit to convert standard basis to Bell basis of two qubits.
- Looks like this



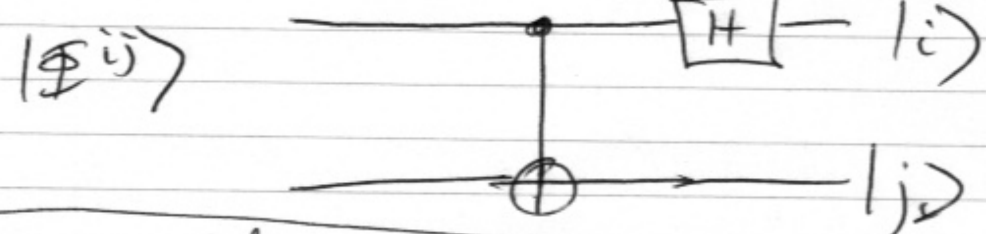

where $|\Phi^{00}\rangle = \dfrac{|00\rangle + |11\rangle}{\sqrt{2}}$

$$|\Phi^{01}\rangle = \frac{|01\rangle + |10\rangle}{\sqrt{2}}$$

$$|\Phi^{10}\rangle = \frac{|00\rangle - |11\rangle}{\sqrt{2}}$$

$$|\Phi^{11}\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}}$$

so then reverse circuit transforms Bell basis to standard basis:

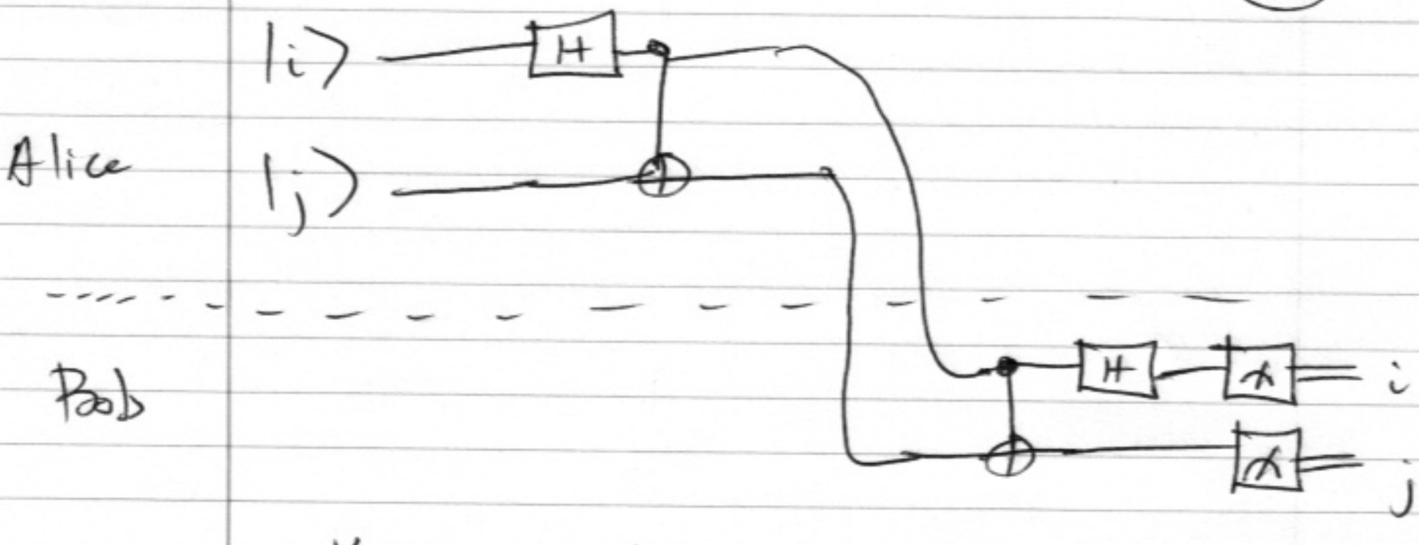



Now, consider

Key observation:

$$|\Phi^{ij}\rangle = (X^j Z^i \otimes I)|\Phi^{00}\rangle$$
$$= (X^j Z^i \otimes I)|\Phi^{+}\rangle$$

this means that Alice (sender) can communicate two bits to Bob (receiver) as follows:



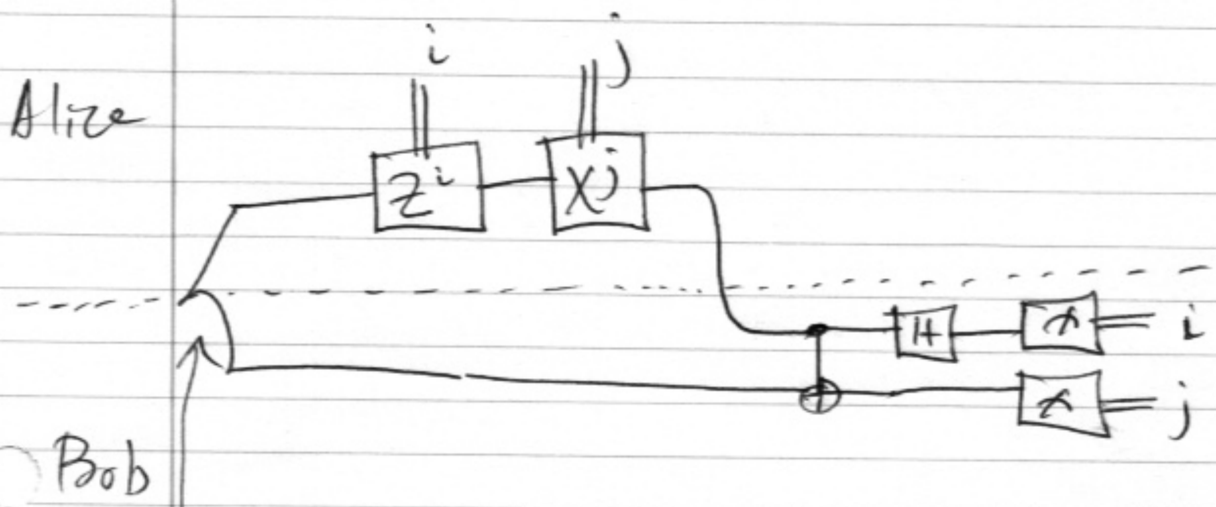

this use two qubit channels to send two bits.

Is it possible to do better?

Yes! use key observation.

It indicates that any of the four Bell states can be reached from the starting state $|\Phi^+\rangle$ by means of a local transformation.

This means the protocol can be modified as follows:




$|\Phi^+\rangle$

idea is that entangled state is distributed beforehand (during off peak times in a network)

Then Alice does local operations + sends one qubit to Bob,

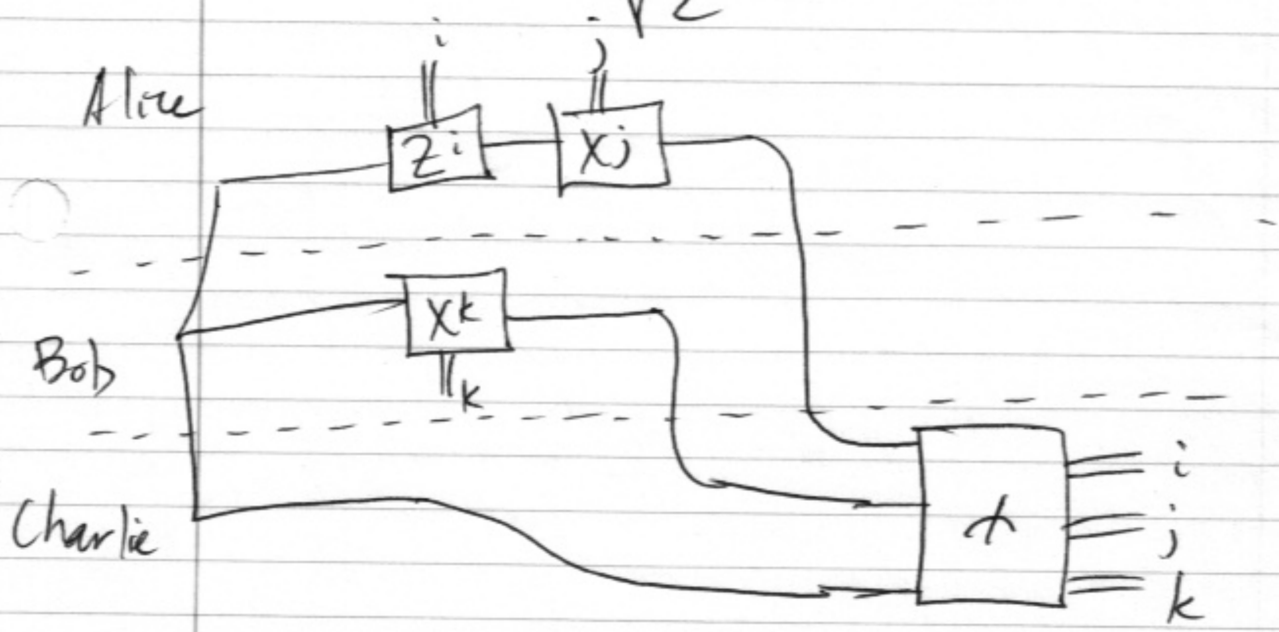
So the protocol can be viewed as the following resource conversion:

1 qubit + 1 ebit $\rightarrow$ 2 classical bits

__Exercise:__

Consider the following variation of superdense coding involving a GHZ state

$$\frac{|000\rangle + |111\rangle}{\sqrt{2}}$$

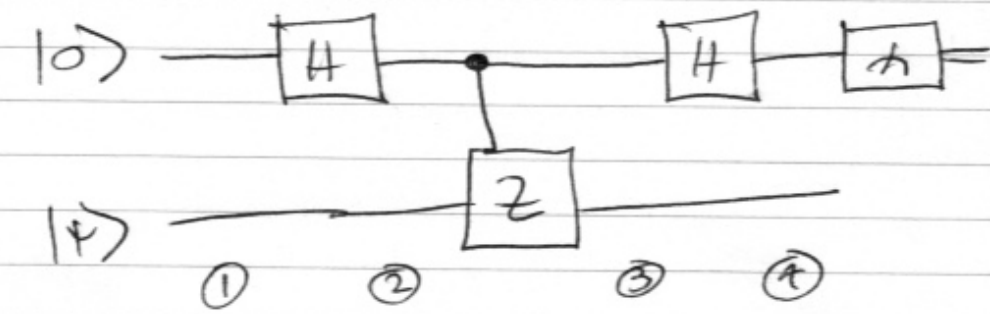



Which circuit to use here?

(to get i, j, k)

What circuit encodes GHZ state?

Let's look at some basic q. circuits:




idea is to track the overall state as it progresses through the circuit

Remember:

$$H = |+\rangle\langle 0| + |-\rangle\langle 1|$$

$$\text{C-}Z = |0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes Z$$

① $|0\rangle|\psi\rangle$

$\rightarrow$ ② $\left(\frac{|0\rangle+|1\rangle}{\sqrt{2}}\right)|\psi\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle|\psi\rangle + |1\rangle|\psi\rangle\right)$

$\rightarrow$ ③ $\frac{1}{\sqrt{2}}\left(|0\rangle|\psi\rangle + |1\rangle Z|\psi\rangle\right)$

$\rightarrow$ ④ $\frac{1}{\sqrt{2}}\left(|+\rangle|\psi\rangle + |-\rangle Z|\psi\rangle\right)$ (*)

After measurement, two questions:

1) What is the probability of obtaining each outcome?

2) What is the post-measurement state in each case?

To answer 1), remember

Born rule: Given $\{\Pi_m\}_m$ such that $\sum_m \Pi_m = I$, probability to obtain outcome $m$ is $p(m) = \| \Pi_m |\psi\rangle \|_2^2$ & post-measurement state is then

$$\frac{\Pi_m |\psi\rangle}{\sqrt{p(m)}}$$

In this case, what is $\Pi_m$?

what is $|\varphi\rangle$?

we calculated that $|\varphi\rangle$ is $(*)$

$\{\Pi_m\}_m$ in this case is

$$\{|0\rangle\langle 0| \otimes I,\ |1\rangle\langle 1| \otimes I\}$$

legitimate measurement b/c

both projections + sum is $I \otimes I$

Then $p(0)$ is

$$p(0) = \left\| \left(|0\rangle\langle 0| \otimes I\right) \frac{1}{\sqrt{2}} \left( |+\rangle|\psi\rangle + |-\rangle Z|\psi\rangle \right) \right\|_2^2$$

$$= \left\| |0\rangle \left( \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} |\psi\rangle + \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}} Z|\psi\rangle \right\|_2^2$$

$$= \left\| \frac{1}{2} |0\rangle (I + Z)|\psi\rangle \right\|_2^2$$

$$\frac{I+Z}{2} = |0\rangle\langle 0|$$

$$\Rightarrow p(0) = \| \, |0\rangle \otimes |0\rangle\langle 0|\psi\rangle \|_2^2$$

$$= |\langle 0|\psi\rangle|^2$$

whole procedure realizes a measurement of $Z = |0\rangle\langle 0| - |1\rangle\langle 1|$

What is post-measurement state?

$$\frac{\Pi_m |\psi\rangle}{\sqrt{p(m)}}$$

$$= \frac{|0\rangle \otimes |0\rangle \langle 0|\psi\rangle}{|\langle 0|\psi\rangle|}$$

Writing $\langle 0|\psi\rangle = re^{i\theta}$ gives

$$= e^{i\theta}|0\rangle \otimes |0\rangle$$

global phase

$$\langle\psi|X|\psi\rangle = p(0) - p(1)$$

$$= |\langle +|\psi\rangle|^2 - |\langle -|\psi\rangle|^2$$

$$= \langle\psi|+\rangle\langle +|\psi\rangle - \langle\psi|-\rangle\langle -|\psi\rangle$$

$$= \langle\psi|\Big(\underbrace{|+\rangle\langle +| - |-\rangle\langle -|}_{X}\Big)|\psi\rangle$$

for Y circuit,

$$p(0) = |\langle +_Y|\psi\rangle|^2$$

post-meas. state $|0\rangle|+_Y\rangle$

$$p(1) = |\langle -_Y|\psi\rangle|^2$$

post-meas. state

$$|1\rangle|-_Y\rangle$$

$$\langle\psi|Y|\psi\rangle = \langle\psi|\Big(|+_Y\rangle\langle +_Y| - |-_Y\rangle\langle -_Y|\Big)|\psi\rangle$$

$$= |\langle +_Y|\psi\rangle|^2 - |\langle -_Y|\psi\rangle|^2$$

$$= p(0) - p(1)$$

What about other possibility?

$$p(1) = \left\| \left( |1\rangle\langle 1| \otimes I \right) \frac{1}{\sqrt{2}} \left( |+\rangle|\psi\rangle + |-\rangle Z|\psi\rangle \right) \right\|_2^2$$

$$= \left\| \frac{1}{\sqrt{2}} |1\rangle \otimes \left( \frac{1}{\sqrt{2}} |\psi\rangle - \frac{1}{\sqrt{2}} Z|\psi\rangle \right) \right\|_2^2$$

$$= \left\| \frac{1}{2} |1\rangle \otimes (I - Z) |\psi\rangle \right\|_2^2$$

$$\Rightarrow \quad \frac{I - Z}{2} = |1\rangle\langle 1|$$

$$= \left\| |1\rangle \otimes |1\rangle\langle 1|\psi\rangle \right\|_2^2$$

$$= |\langle 1|\psi\rangle|^2$$

post-measurement state is

$$\frac{\Pi_m|\psi\rangle}{\sqrt{p(m)}} = \frac{|1\rangle \otimes |1\rangle \langle 1|\psi\rangle}{|\langle 1|\psi\rangle|}$$

Write $\langle 1|\psi\rangle = r_1 e^{i\theta_1}$

$\Rightarrow$

$= e^{i\theta_1} |1\rangle \otimes |1\rangle$

Now recall that

$Z = |0\rangle\langle 0| - |1\rangle\langle 1|$

Let us define a random variable

$V$ that takes value $+1$

w/ probability $p(0)$

& $-1$ w/ prob. $p(1)$

then $\mathbb{E}[V] = (+1)\, p(0) + (-1)\, p(1)$

$= p(0) - p(1)$

$= |\langle 0|\psi\rangle|^2 - |\langle 1|\psi\rangle|^2$

$$= \langle\psi|0\rangle\langle 0|\psi\rangle - \langle\psi|1\rangle\langle 1|\psi\rangle$$

$$= \langle\psi|\Big(|0\rangle\langle 0| - |1\rangle\langle 1|\Big)|\psi\rangle$$

$$= \underbrace{\langle\psi|Z|\psi\rangle}$$

this is an important formula called expectation of Z observable for state $|\psi\rangle$.

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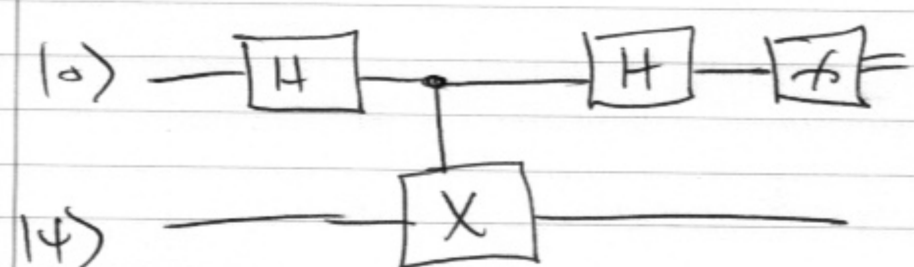
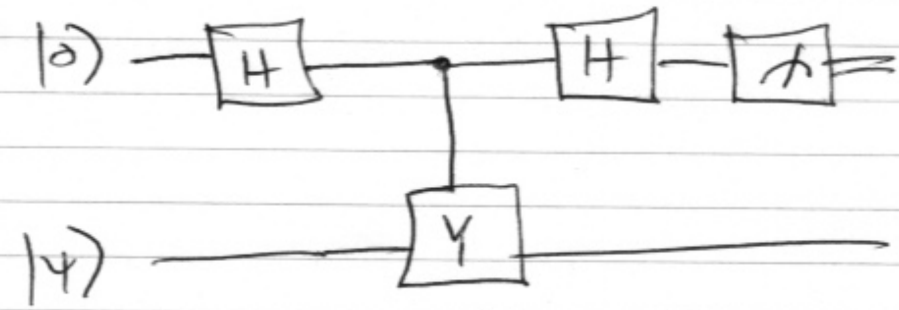
We can repeat the whole exercise for Pauli X & Y w/ circuits






for X circuit, probability to get outcome 0 is

$$p(0) = |\langle + | \psi \rangle|^2 \quad \&$$

post-measurement state is

$$|0\rangle |+\rangle$$

& probability to get outcome 1 is

$$p(1) = |\langle - | \psi \rangle|^2 \quad \&$$

post-meas. state is

$$|1\rangle |-\rangle$$

some comments:

global phases have no observable effect on experiments. This follows from the measurement postulate:

$$p(m) = \| \Pi_m |\psi\rangle \|_2^2$$
$$= \| \Pi_m e^{i\varphi} |\psi\rangle \|_2^2$$

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in super-dense coding protocol, an eavesdropper cannot figure out anything about messages $i,j$ by intercepting qubit transmitted.

Model eavesdropper's measurement by observable $M$.

then we find that

$$\langle \Phi^{ij} | M \otimes I | \Phi^{ij} \rangle$$

$$= \langle \Phi^+ | (I \otimes X^j Z^i)(M \otimes I) \cancel{(X^j Z^i \otimes I)|\Phi^+\rangle}$$
$$(I \otimes Z^i X^j)|\Phi^+\rangle$$

$$= \langle \Phi^+ | M \otimes X^j Z^i Z^i X^j | \Phi^+ \rangle$$

$$= \langle \Phi^+ | M \otimes I | \Phi^+ \rangle$$

$$= \frac{1}{2} \sum_{i,j} \langle i | \langle i | (M \otimes I) | j \rangle | j \rangle$$

$$= \frac{1}{2} \sum_{ij} \langle i | M | j \rangle \langle i | j \rangle$$

$$= \frac{1}{2} \sum_{i} \langle i | M | i \rangle = \frac{1}{2} \mathrm{Tr}[M]$$

independent of $i + j$