Lecture 5

Recall tensor product

$A \otimes B$ (matrices)

$|\psi\rangle \otimes |\phi\rangle$ (vectors)

needed to describe state of multiple q. systems.

obeys properties:

1) for $z \in \mathbb{C}$,

$$z\left(|\psi\rangle \otimes |\phi\rangle\right) = \left(z|\psi\rangle\right) \otimes |\phi\rangle$$
$$= |\psi\rangle \otimes \left(z|\phi\rangle\right)$$

2) distributive properties

$$\left(|\psi_1\rangle + |\psi_2\rangle\right) \otimes |\phi\rangle$$
$$= |\psi_1\rangle \otimes |\phi\rangle + |\psi_2\rangle \otimes |\phi\rangle$$

$$|\psi\rangle \otimes \left(|\phi_1\rangle + |\phi_2\rangle \right)$$

$$= |\psi\rangle \otimes |\phi_1\rangle + |\psi\rangle \otimes |\phi_2\rangle$$

As discussed last time,

$$(A \otimes B)\left(|\psi\rangle \otimes |\phi\rangle \right)$$

$$= A|\psi\rangle \otimes B|\phi\rangle$$

with interpretation

We can write a general $|\varphi\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2$

as $$|\varphi\rangle = \sum_i c_i |\psi_i\rangle \otimes |\phi_i\rangle$$

$\Rightarrow$

$$(A \otimes B)|\varphi\rangle =$$

$$(A \otimes B) \sum_i c_i |\psi_i\rangle \otimes |\phi_i\rangle$$

$$= \sum_i c_i \, A|\psi_i\rangle \otimes B|\phi_i\rangle$$

A general linear operator

$$C: H_1 \otimes H_2 \to H_1 \otimes H_2$$

can be written as

$$C = \sum_i d_i A_i \otimes B_i$$

Then

$$C\,|\psi\rangle \otimes |\phi\rangle$$

$$= \left(\sum_i d_i A_i \otimes B_i\right)\left(|\psi\rangle \otimes |\phi\rangle\right)$$

$$= \sum_i d_i A_i |\psi\rangle \otimes B_i |\phi\rangle$$

What about inner products?

Consider that

$$\left(\langle\psi_1| \otimes \langle\phi_1|\right)\left(|\psi_2\rangle \otimes |\phi_2\rangle\right)$$
$$= \langle\psi_1|\psi_2\rangle \langle\phi_1|\phi_2\rangle$$

(can check this carefully,
but it is a basic rule
we will always use.)

Then defining

$$|\varphi_1\rangle = \sum_i \alpha_i \, |\psi_i\rangle \otimes |\phi_i\rangle \ \&$$

$$|\varphi_2\rangle = \sum_j \beta_j \, |\psi_j\rangle \otimes |\phi_j\rangle$$

we have

$$\langle \varphi_1 | \varphi_2 \rangle$$

$$= \left(\sum_i \alpha_i^* \langle \psi_i | \otimes \langle \phi_i | \right) \left(\sum_j \beta_j \, |\psi_j\rangle \otimes |\phi_j\rangle \right)$$

$$= \sum_{ij} \alpha_i^* \beta_j \Big(\langle \psi_i | \otimes \langle \phi_i | \Big) \Big(|\psi_j\rangle \otimes |\phi_j\rangle \Big)$$

$$= \sum_{ij} \alpha_i^* \beta_j \langle \psi_i | \psi_j \rangle \langle \phi_i | \phi_j \rangle .$$

can form an orthonormal basis for an TP Hilbert space by using O.N. bases for individual spaces, I.e.,

given $\{|i\rangle_A\}_i$ & $\{|j\rangle_B\}_j$

then $\{|i\rangle_A \otimes |j\rangle_B\}_{i,j}$ is an O.N. basis

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Important O.N. basis for teleportation & super-dense coding: Bell Basis

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$$

$$|\Phi^-\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle - |11\rangle\right)$$

$$|\Psi^+\rangle = \frac{1}{\sqrt{2}}\left(|01\rangle + |10\rangle\right)$$

$$|\Psi^-\rangle = \frac{1}{\sqrt{2}}\left(|01\rangle - |10\rangle\right)$$

Exercise: prove orthonormal basis.

Also, prove that, for arbitrary $|\psi\rangle$,

$$\left(\langle\psi|_A \otimes I_B\right)|\Phi^+\rangle_{AB} = |\bar{\psi}\rangle_B$$

complex conjugate

key idea behind "remote state preparation".

Another exercise: Consider that overlaps of $\{|0\rangle, |1\rangle, |+\rangle, |-\rangle\}$ are either $1, 0, \frac{1}{\sqrt{2}}$, or $-\frac{1}{\sqrt{2}}$

What are overlaps of $\{|0\rangle^{\otimes n}, |1\rangle^{\otimes n}, |+\rangle^{\otimes n}, |-\rangle^{\otimes n}\}$?

Basic qubit matrices are Pauli matrices

$$\sigma_0 = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \sigma_1 = \sigma_X = X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

$$\sigma_2 = \sigma_Y = Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \sigma_3 = \sigma_Z = Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

All are Hermitian & unitary, thus eigenvalues are ± 1.

Any 2×2 matrix can be written as a linear combination of Paulis. I.e.,

$$A = \sum_{i \in \{0, \ldots, 3\}} \alpha_i \sigma_i$$

where $\alpha_i \in \mathbb{C}$.

This is related to the fact that Paulis form an O.N. basis w/r.t. Hilbert-Schmidt inner product

$$\langle \sigma_i, \sigma_j \rangle \equiv \mathrm{Tr}[\sigma_i^\dagger \sigma_j]$$
$$= \mathrm{Tr}[\sigma_i \sigma_j] = 2\,\delta_{ij}$$

Then follows that a $2^n \times 2^n$ matrix C can be written in terms of ~~[illegible]~~ tensor products of Paulis as

$$C = \sum_{i_1,\ldots,i_n \in \{0,\ldots,3\}} \gamma_{i_1,\ldots,i_n}\, \sigma_{i_1} \otimes \cdots \otimes \sigma_{i_n}$$

Functions of matrices

Given Hermitian matrix A,

w/ spec. decomp.

$$A = \sum_i a_i |\phi_i\rangle\langle\phi_i| ,$$

we set

$$f(A) = \sum_i f(a_i) |\phi_i\rangle\langle\phi_i|$$

(comes from Taylor series)

Then, for example,

$$\exp(i\theta Z) = \begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix}$$

We also use trace of a square matrix (a lot)

$$Tr\{A\} = \sum_i A_{ii}$$ (sum of diagonal elements)

independent of O.N. basis used, so that

$$\mathrm{Tr}\{A\} = \sum_i \langle \phi_i | A | \phi_i \rangle$$

where $\{|\phi_i\rangle\}_{\S i}$ is an arbitrary O.N. basis.

Trace has a cyclic property:

$$\mathrm{Tr}\{ABC\} = \mathrm{Tr}\{CAB\} = \mathrm{Tr}\{BCA\}$$

$$\Rightarrow \mathrm{Tr}\{UAU^+\} = \mathrm{Tr}\{U^+UA\}$$
$$= \mathrm{Tr}\{A\}.$$

can also show that

$$\underbrace{\mathrm{Tr}\{A\,|\psi\rangle\langle\psi|\} = \langle\psi|A|\psi\rangle}_{\text{basic equations of Q.M.}}$$

Postulates of Q.M.

I. Associated to an isolated physical system is a Hilbert space.
State of the system described by $\underbrace{\text{state vector}}_{\text{used to predict outcomes of experiments}}$ (unit vector)

II. Evolution of closed system described by unitary transformation

$$|\psi'\rangle = U|\psi\rangle$$

(note that unitaries ~~p~~preserve the norm)

III. Basic kind of measurement is projective measurement $\{\Pi_m\}_m$ (such that $\Pi_m^2 = \Pi_m$, Π_m Hermitian)

$$\sum_m \Pi_m = I$$

probability of getting outcome m,
after performing measurement on state $|\psi\rangle$,
is $p(m) = \| \Pi_m |\psi\rangle \|_2^2 = \langle \psi | \Pi_m | \psi \rangle$

post-meas. state is

$$\frac{\Pi_m |\psi\rangle}{\sqrt{p(m)}}.$$ (can check this is a state)

IV. state space of composite system is a tensor-product Hilbert space.

if n systems have not interacted & are each prepared as $|\psi_i\rangle$, then joint state is

$$|\psi_1\rangle \otimes |\psi_2\rangle \otimes \cdots \otimes |\psi_n\rangle$$