ECE 4950 Fall 2023
Quantum Information Science: Communication and Computation
Homework 3

Due Monday 9 October 2023, by 8:40am in class (no late homeworks accepted)

(You are allowed to work with 1-2 other classmates as long as you write down who your collaborators are.)

1. In class, we discussed a quantum algorithm for estimating the quantity $\mathrm{Re}[\langle\psi|U|\psi\rangle]$, where the unitary U is a $d \times d$ dimensional matrix, when given access to controlled-U and the d-dimensional state $|\psi\rangle$.

   (a) As a generalization of this algorithm, develop a quantum algorithm that estimates the quantity $\mathrm{Re}[\mathrm{Tr}[U]]/d$. (Hint: Recall the definition of the trace of a matrix and consider picking the initial state uniformly at random from the standard basis.)

   (b) As another generalization of this algorithm, suppose that you are given access to controlled-U_1, controlled-U_2, and the state $|\psi\rangle$. Design a quantum algorithm that estimates the quantity $\mathrm{Re}[\langle\psi|\,\mathrm{Re}[U_2]U_1|\psi\rangle]$, where $\mathrm{Re}[U_2] := \frac{1}{2}\left(U_2 + U_2^\dagger\right)$. Give a circuit diagram for the algorithm, calculate the probabilities for all measurement outcomes, and show the classical postprocessing needed to arrive at the conclusion that the algorithm estimates $\mathrm{Re}[\langle\psi|\,\mathrm{Re}[U_2]U_1|\psi\rangle]$ correctly. (Hint: Use two control qubits and assign a random variable Z the value $+1$ if the measurement outcomes of the control qubits are equal and set it to -1 otherwise.)

2. The goal of this exercise is to design a higher-dimensional version of super-dense coding, beyond the qubit case discussed in class. Define the Heisenberg–Weyl operators in the following way:

$$X(x) := \sum_{k=0}^{d-1} |(k+x) \bmod d\rangle\langle k|, \tag{1}$$

$$Z(z) := \sum_{k=0}^{d-1} e^{2\pi i z k/d}|k\rangle\langle k|, \tag{2}$$

   where $x, z \in \{0, \ldots, d-1\}$ and $\{|k\rangle\}_{k=0}^{d-1}$ is an orthonormal basis. We will first go through some properties of these operators and then think about the higher-dimensional version of super-dense coding.

   (a) Show that these operators reduce to Pauli X and Z when $d = 2$.

   (b) Show that the inverse of $X(x)$ is $X(-x)$. Show that the inverse of $Z(z)$ is $Z(-z)$.

   (c) Show that $X(x)Z(z) = e^{-2\pi i z x/d} Z(z)X(x)$.

   (d) Define the maximally entangled state of d dimensions as

$$|\Phi_d\rangle_{AB} := \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} |j\rangle_A |j\rangle_B. \tag{3}$$

Show that $\{|\Phi^{x,z}\rangle_{AB}\}_{x,z}$ is an orthonormal basis, where

$$|\Phi^{x,z}\rangle_{AB} \coloneqq (X(x)Z(z) \otimes I)|\Phi_d\rangle_{AB}. \tag{4}$$

(e) Design a d-dimensional super-dense coding protocol. That is, suppose that Alice and Bob share the state $|\Phi_d\rangle_{AB}$ in advance and can use an ideal d-dimensional quantum channel from Alice to Bob. Then show how Alice can encode one of d^2 possible messages into $|\Phi_d\rangle_{AB}$, while Bob can recover the message perfectly after they use the ideal quantum channel.

3. Multiparty super-dense coding. Another generalization of super-dense coding involves mutiple parties.

   (a) A GHZ three-party entangled state is defined by $|\Phi_{\text{GHZ}}\rangle_{ABC} \coloneqq \frac{1}{\sqrt{2}}(|000\rangle_{ABC} + |111\rangle_{ABC})$. Write down a quantum circuit that generates this state, starting from the initial state $|000\rangle_{ABC}$.

   (b) Show that

$$\frac{1}{\sqrt{2}}(|000\rangle_{ABC} \pm |111\rangle_{ABC}), \tag{5}$$
$$\frac{1}{\sqrt{2}}(|001\rangle_{ABC} \pm |110\rangle_{ABC}), \tag{6}$$
$$\frac{1}{\sqrt{2}}(|010\rangle_{ABC} \pm |101\rangle_{ABC}), \tag{7}$$
$$\frac{1}{\sqrt{2}}(|100\rangle_{ABC} \pm |011\rangle_{ABC}), \tag{8}$$

   is an orthonormal basis.

   (c) Design a multi-party super-dense coding protocol. That is, suppose that Alice, Bob, and Charlie are spatially separated and share the GHZ state $|\Phi_{\text{GHZ}}\rangle_{ABC}$ before communication begins. What operations should Alice and Bob perform such that Alice can send two bits to Charlie and Bob can send one bit to Charlie, by making use of the state $|\Phi_{\text{GHZ}}\rangle_{ABC}$ and ideal qubit channels from Alice to Charlie and from Bob to Charlie?

4. Exercises 6.2.3, 6.2.4, 6.2.7 in arXiv:1106.1445.