ECE 4950 Fall 2023
Quantum Information Science: Communication and Computation
Homework 2

Due Monday 25 September 2023, by 8:40am in class (no late homeworks accepted)

(You are allowed to work with 1-2 other classmates as long as you write down who your collaborators are.)

1. The singular value decomposition of an $m \times n$ matrix A can be written as $A = U\Sigma V^\dagger$, where U is an $m \times m$ unitary matrix, Σ is an $m \times n$ rectangular, diagonal matrix containing r singular values (where r is the rank of A), and V is an $n \times n$ unitary matrix. Show that this is equivalent to writing A as

$$A = \sum_{i=1}^{r} s_i |\phi_i\rangle\langle\psi_i|, \tag{1}$$

where $\{s_i\}_{i=1}^r$ is a set of strictly positive singular values, $\{|\phi_i\rangle\}_{i=1}^m$ is an orthonormal basis, and $\{|\psi_i\rangle\}_{i=1}^n$ is an orthonormal basis.

2. Define the SWAP operator as

$$\text{SWAP} \coloneqq \sum_{i,j=1}^{d} |i\rangle\langle j| \otimes |j\rangle\langle i|. \tag{2}$$

(a) Show that this indeed performs a swap, in the following sense:

$$\text{SWAP}(|\psi\rangle \otimes |\phi\rangle) = |\phi\rangle \otimes |\psi\rangle. \tag{3}$$

(b) Show that SWAP is Hermitian and unitary.

(c) Show that the eigenvalues of SWAP are $+1$ and -1. Show that the projection onto the $+1$-eigenspace of SWAP is $\frac{1}{2}(I \otimes I + \text{SWAP})$, and show that the projection onto the -1-eigenspace of SWAP is $\frac{1}{2}(I \otimes I - \text{SWAP})$.

(d) For $d = 2$, show that

$$\text{SWAP} = |\Phi^+\rangle\langle\Phi^+| + |\Phi^-\rangle\langle\Phi^-| + |\Psi^+\rangle\langle\Psi^+| - |\Psi^-\rangle\langle\Psi^-|, \tag{4}$$

where these are the Bell states. Conclude that

$$\frac{1}{2}(I \otimes I + \text{SWAP}) = |\Phi^+\rangle\langle\Phi^+| + |\Phi^-\rangle\langle\Phi^-| + |\Psi^+\rangle\langle\Psi^+|, \tag{5}$$

$$\frac{1}{2}(I \otimes I - \text{SWAP}) = |\Psi^-\rangle\langle\Psi^-|, \tag{6}$$

in this case.

(e) Show that, for $d \times d$ matrices A and B,

$$\text{SWAP}(A \otimes B)\text{SWAP}^\dagger = B \otimes A, \tag{7}$$

again demonstrating the swapping properties of SWAP.

(f) Show that

$$\mathrm{Tr}[\mathrm{SWAP}(A \otimes B)] = \mathrm{Tr}[AB]. \tag{8}$$

(g) Define the partial transpose operation as

$$(\mathrm{id}_1 \otimes T_2)(W_{12}) = \sum_{i,j} (I_1 \otimes |i\rangle\langle j|_2)(W_{12})(I_1 \otimes |i\rangle\langle j|_2). \tag{9}$$

Show that

$$(\mathrm{id}_1 \otimes T_2)\big(\Phi^d\big) = \frac{1}{d}\mathrm{SWAP}, \tag{10}$$

where Φ^d is the maximally entangled state, defined as

$$\Phi^d := |\Phi^d\rangle\langle\Phi^d|, \tag{11}$$

$$|\Phi^d\rangle := \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |i\rangle|i\rangle. \tag{12}$$

3. Define the cyclic shift operation as

$$\mathrm{CYC}_n := \sum_{i_1, i_2, \ldots, i_n} |i_1\rangle\langle i_2| \otimes |i_2\rangle\langle i_3| \otimes \cdots \otimes |i_{n-1}\rangle\langle i_n| \otimes |i_n\rangle\langle i_1|. \tag{13}$$

(a) Prove that the cyclic shift is unitary. Is it Hermitian?

(b) For $d \times d$ matrices $A_1, \ldots, A_n$, prove that

$$\mathrm{Tr}[\mathrm{CYC}_n(A_1 \otimes \cdots \otimes A_n)] = \mathrm{Tr}[A_1 \cdots A_n]. \tag{14}$$

4. Tricks with maximally entangled states:

(a) Prove the transpose trick. That is, for a matrix A, prove that the following holds:

$$(A \otimes I)|\Phi^d\rangle = \big(I \otimes A^T\big)|\Phi^d\rangle. \tag{15}$$

(b) Prove that

$$\big(I_1 \otimes \langle\Phi^d|_{23} \otimes \langle\Phi^d|_{45} \otimes \langle\Phi^d|_{67}\big)\big(|\Phi^d\rangle_{12} \otimes |\Phi^d\rangle_{34} \otimes |\Phi^d\rangle_{56} \otimes I_7\big) = \frac{1}{d^3} I_{7\to 1}, \tag{16}$$

where $I_{7\to 1}$ is the identity operator that takes system 7 to system 1.

5. Use the Cauchy–Schwarz inequality to prove that the maximally entangled state $|\Phi^+\rangle$ of two qubits cannot be written as a product $|\phi\rangle \otimes |\psi\rangle$, and so it is thus entangled. (Hint: prove that $|\langle\Phi^+|(|\phi\rangle \otimes |\psi\rangle)|^2 \leq 1/2$).

6. Exercises 3.3.11, 3.5.4, 3.5.5, 3.5.6, 3.6.1 in arXiv:1106.1445.