QEC Lecture

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A general quantum error correcting code encodes quantum states into a subspace of a larger Hilbert space. This is accomplished via

1) appending fresh ancillas
2) performing a unitary

The projection onto the codespace plays a distinguished role.

For repetition code,

$$P = |000\rangle\langle 000| + |111\rangle\langle 111|$$

Error correction proceeds in 2 steps

$$\left.\begin{array}{l}\text{1) syndrome measurement}\\ \text{2) recovery}\end{array}\right\}\ \text{can describe these as recovery channel}$$

can describe the effects of noise as a channel $\mathcal{E}$ & recovery as a channel $\mathcal{R}$. For error correction to be successful, we should have that

$$(\mathcal{R} \circ \mathcal{E})(\rho) = \rho$$

$\forall$ states in codespace, i.e.,

$$P \rho P = \rho .$$

where P is codespace projection.

It is of interest to know when a code can correct an error channel $\mathcal{E}$.

The QEC conditions answer this:

<u>Thm</u>: Let C be a code & let P be the codespace projection. Let $\mathcal{E}$ be an error channel w/ Kraus operators $\{E_i\}_i$.
A necessary & sufficient condition for the existence of a recovery channel ~~R~~ R for $\mathcal{E}$ on C is that

$$P E_i^\dagger E_j P = \alpha_{ij} P$$

where α_{ij} is a PSD Hermitian matrix.

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these conditions are equivalent to

$$\langle \bar{k} | E_i^\dagger E_j | \bar{l} \rangle = \alpha_{ij} \, \delta_{kl}$$

where $|\bar{k}\rangle$ & $|\bar{l}\rangle$ are
q. codewords & so codespace
projection is $P = \sum_k' |\bar{k}\rangle\langle\bar{k}|$

Interpretation:

For $k \neq l$ the above condition is necessary. If it did not hold, errors would destroy the perfect distinguishability of orthogonal codewords

If $\alpha_{ij} = \delta_{ij}$, the code is non-degenerate & different errors take codespace to orthogonal & distinguishable subspaces. But this is not necessary.

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1^{st} prove sufficiency

Let $\{E_i\}_i$ be Kraus operators satisfying the QEC conditions.

α is Hermitian PSD & so can be diagonalized as

$$d = u^\dagger \alpha u$$

where d is diagonal & positive entries & u is unitary.

Let $F_k = \sum_i u_{ik} E_i$ &

note that

$$\mathcal{E}(\cdot) = \sum_i E_i (\cdot) E_i^\dagger = \sum_k F_k (\cdot) F_k^\dagger$$

We then find that

$$P F_k^\dagger F_l P = \sum_{ij} u_{ki}^\dagger u_{jl} P E_i^\dagger E_j P$$

use QEC conditions to get that this equals

$$\sum_{ij} u_{ki}^{\dagger} \alpha_{ij} u_{jl} \; P = d_{kl} P$$

So we get that

$$P F_k^{\dagger} F_l P = d_{kl} P$$

(simpler QEC conditions)

We can now define the syndrome measurement:

the effect of the noise op. F_k on codespace P is

$$F_k P = U_k \sqrt{P F_k^{\dagger} F_k P}$$

← polar decomp.

$$= \sqrt{d_{kk}}\, U_k P$$

for some unitary U_k

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noise rotates codespace into subspace defined by projection

$$P_k = \underbrace{U_k P U_k^\dagger}_{\text{rotated codespace}} = \frac{F_k P U_k^\dagger}{\sqrt{d_{kk}}}$$

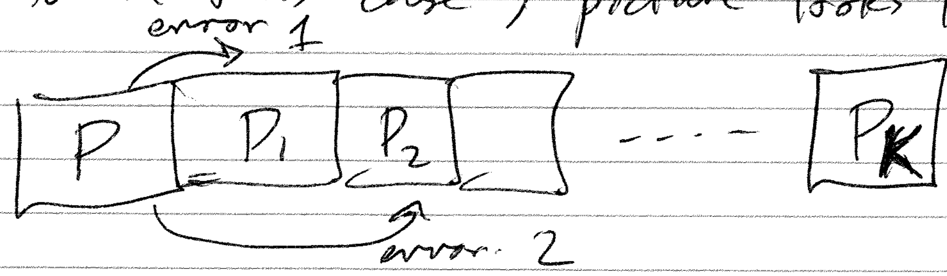
subspaces are orthogonal

b/c for $k \neq l$

$$P_l P_k = P_l^\dagger P_k = \frac{U_l P F_l^\dagger F_k P U_k^\dagger}{\sqrt{d_{ll} d_{kk}}} = 0$$

from QEC conditions

So in this case, picture looks like



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then syndrome measurement is

$\{P_k\}_k$ + add another element if needed

so, measure $\{P_k\}_k$ to figure out which subspace ~~you~~ you're in & then apply $U_k^\dagger$ to go back.

$\mathcal{R}$ is then

$$\mathcal{R}(\sigma) = \sum_k U_k^\dagger P_k \sigma P_k U_k$$

recovery

syndrome measurement

Our goal is to show that

$$(\mathcal{R} \circ \mathcal{E})(\rho) = \rho$$

Consider that

$$U_k^\dagger P_k F_\ell \sqrt{\rho}$$

$$= U_k^\dagger P_k^\dagger F_\ell P \sqrt{\rho}$$

$$= U_k^\dagger U_k P F_k^\dagger F_\ell P \sqrt{\rho} \,/\, \sqrt{d_{kk}}$$

$$= \delta_{k\ell} \sqrt{d_{kk}}\, P \sqrt{\rho}$$

$$= \delta_{k\ell} \sqrt{d_{kk}}\, P \sqrt{\rho}$$

$$\Rightarrow (\mathcal{R} \circ \mathcal{E})(\rho) = \sum_{k\ell} U_k^\dagger P_k F_\ell \rho F_\ell^\dagger P_k U_k$$

$$= \sum_{k\ell} \delta_{k\ell} d_{kk} \rho$$

$$\propto \rho$$

Necessity of QEC conditions

Suppose $\{E_i\}_i$ is a set of perfectly correctable error operators by some recovery channel $\mathcal{R}$ w/ Kraus op's $\{R_k\}_k$.

Then for any state ρ in the codespace, we have that

not necessarily

$$(\mathcal{R} \circ \mathcal{E})(P \rho P) = P \rho P$$

~~for some constant~~

$$= \sum_{ik} R_k E_i P \rho P E_i^\dagger R_k^\dagger = P \rho P$$

Since this holds for all ρ

the q. op. w/ elements $\{R_k E_i P\}_{ik}$

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is indistinguishable from q. op.
w/ element P.
From equivalence of q. op.'s
there exists complex numbers c_{ki}
such that

$$R_k E_i P = c_{ki} P$$

Take adjoint & get

$$P E_j^\dagger R_k^\dagger = c_{kj}^* P \Rightarrow$$

$$P E_j^\dagger R_k^\dagger R_k E_i P = c_{kj}^* c_{ki} P$$

Sum over k to get

$$\sum_k P E_j^\dagger R_k^\dagger R_k E_i P$$

$$= P E_j^\dagger E_i P = \underbrace{\sum_k c_{kj}^* c_{ki}}_{\alpha_{ji}} P$$ is PSD.

Discretization of errors:

Suppose C is a q. code & R is recovery op. for an error channel $\mathcal{E}$ w/ Kraus op.'s $\{E_i\}_i$.

If $\mathcal{F}$ is an error channel w/ Kraus op's given by linear combinations of E_i ($F_j = \sum_i m_{ji} E_i$) then R corrects $\mathcal{F}$.

Proof: QEC conditions for $\{E_i\}_i$

$P E_i^\dagger E_j P = \alpha_{ij} P$

take α_{ij} diagonal d_{kk}

recovery op is $U_k^\dagger P_k$

such that for $\rho = P \rho P$

$$U_k^\dagger P_k E_i \sqrt{\rho} = \delta_{ki} \sqrt{d_{kk}} \sqrt{\rho}$$

Substitute $\quad F_j = \sum_j m_{ji} E_i$

$$U_k^\dagger P_k F_j \sqrt{\rho} = \sum_i m_{ji} \delta_{ki} \sqrt{d_{kk}} \sqrt{\rho}$$

$$= m_{jk} \sqrt{d_{kk}} \sqrt{\rho}$$

so then

$$(R \circ F)(\rho)$$

$$= \sum_{kj} U_k^\dagger P_k F_j \rho F_j^\dagger P_k U_k$$

$$= \sum_{kj} |m_{jk}|^2 d_{kk} \rho$$

$$\propto \rho$$

Thus if code corrects single-qubit Pauli errors, then it corrects arbitrary single-qubit errors.

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QEC is more similar to classical digital EC than it is to classical analog EC.