Lecture 17.4

Now let us discuss using linear combination of unitaries for Hamiltonian simulation (1412.4687)

Suppose that

$$H = \sum_{\ell} \alpha_\ell H_\ell$$

where each H_ℓ is unitary & there is a method available for implementing the unitary.

- Note that any Hamiltonian can be decomposed as an LCU because there exist operator bases.

Goal is to simulate

$U = e^{-iHt}$ to within error ε.

Divide time evolution into r segments of length t/r, i.e.,

$$e^{-iHt} = \left(e^{-iHt/r}\right)^r$$

within each segment, approximate evolution as

$$e^{-iHt/r} \approx \sum_{k=0}^{K} \frac{1}{k!} (-iHt/r)^k$$

to have overall accuracy ε, accuracy needed for each segment is ε/r.

to get this accuracy when

$r \geq \|H\| t$, take

$$K = O\left(\frac{\log(r/\varepsilon)}{\log\log(r/\varepsilon)} \right)$$

overall complexity is $\approx r \cdot K$.

Now substitute $H = \sum_{\ell=1}^{L} \alpha_\ell H_\ell$

into truncated Taylor series

to get

$$\sum_{k=0}^{K} \frac{1}{k!} (-iHt/r)^k$$

$$= \sum_{k=0}^{K} \sum_{\ell_1, \ldots, \ell_k = 1}^{L} \frac{(-it/r)^k}{k!} \alpha_{\ell_1} \cdots \alpha_{\ell_k} H_{\ell_1} \cdots H_{\ell_k}$$

This now has the form of

a linear combination of unitaries as

$$\tilde{U} = \sum_{j=0}^{m-1} \beta_j V_j$$

where $\beta_j > 0$ + each V_j

corresponds to $(-i)^k H_{\ell_1} \cdots H_{\ell_k}$

To simulate the segment, we then do LCU:

prepare the state

$$B|0\rangle = \frac{1}{\sqrt{\|\vec{\beta}\|_1}} \sum_j \sqrt{\beta_j}\, |j\rangle$$

using unitary B.

define $\text{select}(V) = \sum_j |j\rangle\langle j| \otimes V_j$

then do

$$W = (B^\dagger \otimes I)(\text{select}(V))(B \otimes I)$$

to realize

$$W|0\rangle|\psi\rangle = \frac{1}{\|\vec{\beta}\|_1}|0\rangle\tilde{U}|\psi\rangle + \sqrt{1-\frac{1}{\|\vec{\beta}\|_1^2}}\,|\Phi\rangle$$

where $|\Phi\rangle$ orthogonal to 1st term.

applying projector $P = |0\rangle\langle 0| \otimes I$

gives

$$PW|0\rangle|\psi\rangle = \frac{1}{\|\vec{\beta}\|_1}|0\rangle\tilde{U}|\psi\rangle$$

$\tilde{U}$ is not unitary but is close to unitary

can then use a robust version of oblivious amplitude amplification w/ reflection $R = I - 2P$

& amplification $A = -W R W^{\dagger} R W$ 6

$\Rightarrow \| P A |0\rangle|\psi\rangle - |0\rangle U_r |\psi\rangle \| = O(\varepsilon/r)$

to amplify this to an

approximate unitary w/out

prefactor of $\frac{1}{\|\vec{\beta}\|_1}$.

can then realize error

ε/r for each segment,

& overall error of $r \cdot \frac{\varepsilon}{r} = \varepsilon$.

see paper for full complexity

analysis.