Lecture 17.2

Q. Eigenvalue transforms:

- previously, we discussed transformation of a single matrix element.
- Now, extend to transform all eigenvalues of a Hermitian matrix H (Hamiltonian) embedded in a unitary.

(Sps. H is $N \times N$)

Suppose now that we have the unitary U that block encodes H as

$$U = \begin{bmatrix} H & \sqrt{I - H^2} \\ \sqrt{I - H^2} & -H \end{bmatrix}$$

$$U = \sigma_z \otimes H + \sigma_x \otimes \sqrt{I - H^2}$$

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Suppose that H has an eigendecomposition as

$$H = \sum_\lambda \lambda |\lambda\rangle\langle\lambda|$$

Then $$\sqrt{I - H^2} = \sum_\lambda \sqrt{1-\lambda^2}\, |\lambda\rangle\langle\lambda|$$

$\Rightarrow$

$$U = \sigma_z \otimes H + \sigma_x \otimes \sqrt{I - H^2}$$

$$= \sigma_z \otimes \sum_\lambda \lambda |\lambda\rangle\langle\lambda| + \sigma_x \otimes \sum_\lambda \sqrt{1-\lambda^2}\, |\lambda\rangle\langle\lambda|$$

$$= \sum_\lambda \lambda \sigma_z \otimes |\lambda\rangle\langle\lambda| + \sqrt{1-\lambda^2}\, \sigma_x \otimes |\lambda\rangle\langle\lambda|$$

$$= \sum_\lambda \left(\lambda \sigma_z + \sqrt{1-\lambda^2}\, \sigma_x\right) \otimes |\lambda\rangle\langle\lambda|$$

$$= \sum_\lambda R(\lambda) \otimes |\lambda\rangle\langle\lambda|$$

$R(\lambda)$ is ~~rotation~~ matrix from before.

If we switch ordering of tensor product to be as follows, then we get direct sum

$$\sum_{\lambda} |\lambda\rangle\langle\lambda| \otimes R(\lambda)$$

$$= \bigoplus_{\lambda} R(\lambda)$$

We thus have N Bloch spheres & we can rotate them all simultaneously by using a common rotation operator.

Alternatively, we have uncovered N signal unitaries ($\{R(\lambda)\}_{\lambda}$) that can be processed simultaneously by some signal processing unitaries

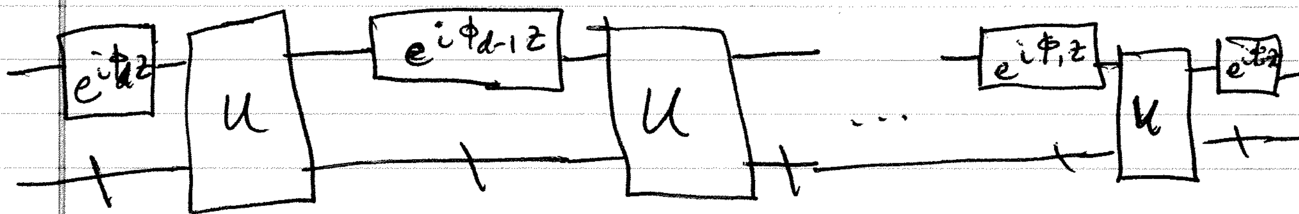
- For the special form of U discussed, we can process all N signals ($\{\lambda\}_\lambda$) simultaneously by using signal processing unitaries of the form $e^{i\phi Z} = S(\phi)$.
- That is, we would do

$$\left(e^{i\phi_0 Z} \otimes I\right) \prod_{k=1}^{d} U \left(e^{i\phi_k Z} \otimes I\right)$$

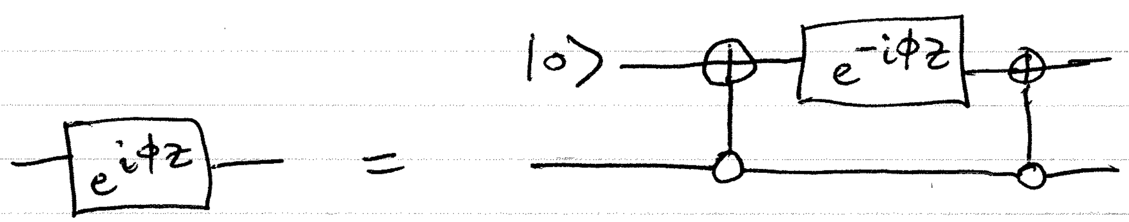
$$= \left(e^{i\phi_0 Z} \otimes I\right) \prod_{k=1}^{d} \left(\sum_\lambda R(\lambda) \otimes |\lambda\rangle\langle\lambda|\right) e^{i\phi_k Z} \otimes I$$

$$= \sum_\lambda \underbrace{e^{i\phi_0 Z}\left(\prod_{k=1}^{d} R(\lambda)\, e^{i\phi_k Z}\right)}_{\text{signal processing of all } N \text{ signals simultaneously.}} \otimes |\lambda\rangle\langle\lambda|$$

Thus, the circuit looks like



An alternative way for realizing rotations is by making use of observation that



Indeed, $e^{i\phi Z}|\psi\rangle = e^{i\phi Z}(\alpha|0\rangle + \beta|1\rangle)$

$$= \alpha e^{i\phi}|0\rangle + \beta e^{-i\phi}|1\rangle$$

& second circuit gives

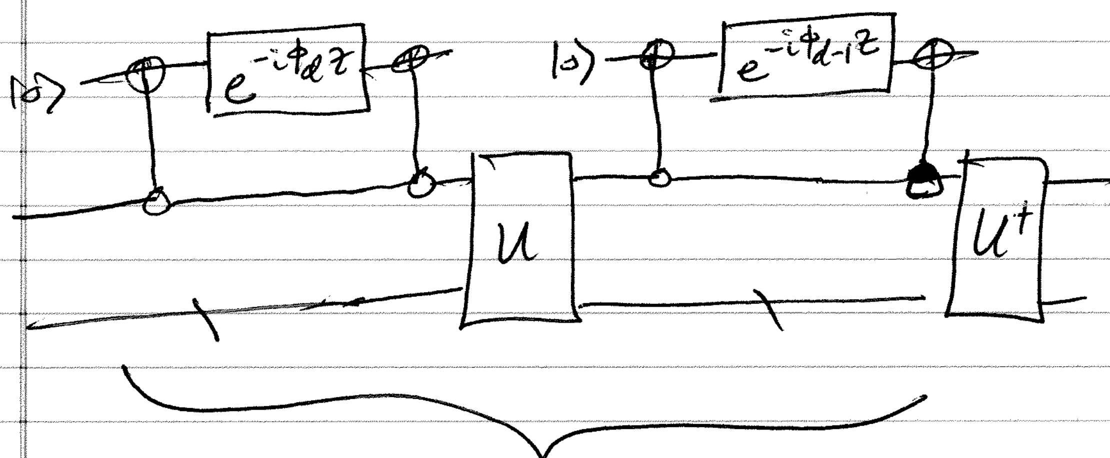
$$|\psi\rangle|0\rangle = \alpha|00\rangle + \beta|10\rangle$$

$$\xrightarrow{\text{CNOT}} \alpha|01\rangle + \beta|10\rangle$$

$$\xrightarrow{Z\text{-rot}} \alpha e^{i\phi}|01\rangle + \beta e^{-i\phi}|10\rangle$$

$$\rightarrow (\alpha e^{i\phi}|0\rangle + \beta e^{-i\phi}|1\rangle)|0\rangle$$

In this case, observe that $U = U^\dagger$, so that circuit can be written as



repeat $d/2$ more times (assuming d is even)

Now identifying $\Pi = |0\rangle\langle 0| \otimes I$

we can write

$$e^{i\phi\sigma_z} \otimes I = e^{i\phi(|0\rangle\langle 0| - |1\rangle\langle 1|)} \otimes I$$

$$= e^{i\phi(|0\rangle\langle 0| \otimes I - |1\rangle\langle 1| \otimes I)}$$

$$= e^{i\phi(\Pi - (I - \Pi))}$$

$$= e^{i\phi(2\Pi - I)}$$

this form is useful for generalizations

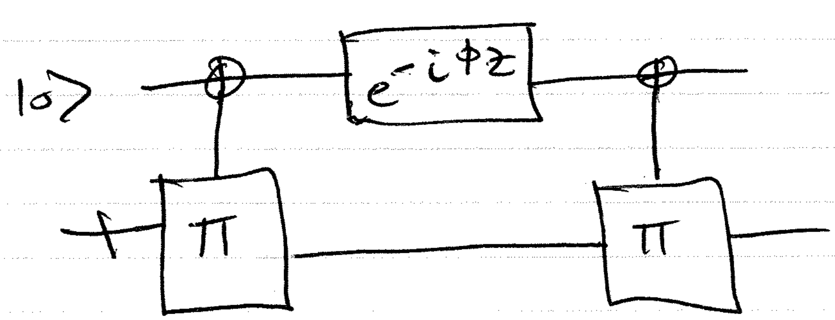
in which U does not have a special form. Suppose instead that

$$U = \begin{bmatrix} H & \cdot \\ \cdot & \cdot \end{bmatrix}$$

where other entries are arbitrary.

Now let Π denote the projection onto the top left block.

Then $e^{i\phi(2\Pi - I)} = \Pi_\phi$ is the appropriate generalization & can be realized by



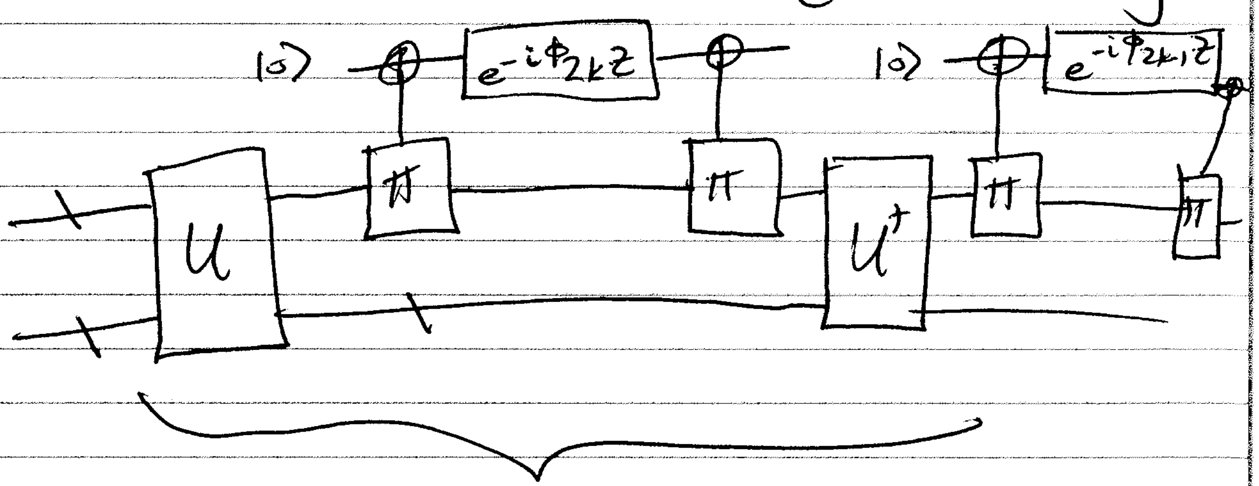
where Π-controlled-Not is

$$\Pi CNOT = \Pi \otimes X + (I - \Pi) \otimes I$$

flip the last qubit if 1st register is in subspace onto which Π projects

Most general unitary circuit for even d looks like

$$\prod_{k=1}^{d/2} \Pi_{\phi_{2k-1}} U^{\dagger} \Pi_{\phi_{2k}} U = \begin{bmatrix} \mathrm{Poly}(H) & \cdot \\ \cdot & \cdot \end{bmatrix}$$



repeat $d/2$ times.

Can then use polynomial transformations for a variety of purposes

One example: eigenvalue filtering

- take eigenvalues above / below a threshold to one & others to zero.

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q. singular value transformation

q. signal processing sequences can be used to transform all the singular values of a rectangular matrix A.

Recall SVD of a matrix A:

$$A = W \Sigma V^{\dagger}$$

where W & V are unitaries

& Σ is a diagonal, rectangular matrix of singular values.

denote columns of W by $\{|w_k\rangle\}_k$
& " " V by $\{|v_k\rangle\}_k$

each orthonormal bases.

can rewrite SVD as

$$A = \sum_{k=1}^{n} \sigma_k |w_k\rangle\langle v_k|$$

Given unitary U such that
A is block encoded as

$$U = \begin{array}{cc} & \Pi \\ \tilde{\Pi} & \begin{bmatrix} A & \cdot \\ \cdot & \cdot \end{bmatrix} \end{array}$$

where $\tilde{\Pi} = \sum_k |w_k\rangle\langle w_k|$ &

$$\Pi = \sum_k |v_k\rangle\langle v_k|$$

these are projectors that locate A w/in U as

$$A = \tilde{\Pi} U \Pi$$

Going forward, for simplicity, let us suppose ~~that~~ A is square & U has form

$$U = \begin{bmatrix} A & \sqrt{I - A^2} \\ \sqrt{I - A^2} & -A \end{bmatrix}$$

& $\sqrt{I - A^2} = \sum_k \sqrt{1 - \sigma_k^2}\, |w_k\rangle\langle v_k|$

can verify that $U^\dagger U = I$

can again write U as

$$U = \sigma_z \otimes A + \sigma_x \otimes \sqrt{I - A^2}$$

$$= \sigma_z \otimes \sum_k \sigma_k \otimes |w_k\rangle\langle v_k| +$$

$$\sigma_x \otimes \sum_k \sqrt{1-\sigma_k^2} \otimes |w_k\rangle\langle v_k|$$

$$= \sum_k \left(\sigma_k \sigma_z + \sqrt{1-\sigma_k^2}\,\sigma_x\right) \otimes |w_k\rangle\langle v_k|$$

$$= \sum_k R(\sigma_k) \otimes |w_k\rangle\langle v_k|$$

&

$$U^\dagger = \sum_k R(\sigma_k) \otimes |v_k\rangle\langle w_k|$$

can then realize a q. signal processing of singular values as

$$\left(e^{i\phi_0 Z} \otimes I\right) U \prod_{k=1}^{(d-1)/2} \left(e^{i\phi_{2k} Z} \otimes I\right) U^\dagger \left(e^{i\phi_{2k+1} Z} \otimes I\right) U$$

$$= \begin{bmatrix} \mathrm{Poly}^{(SV)}(A) & \cdot \\ \cdot & \cdot \end{bmatrix}$$

where $\mathrm{Poly}^{(SV)}(A)$

$$= \sum_k \mathrm{Poly}(\sigma_k)\, |w_k\rangle\langle v_k|$$

More generally, transformation for odd d is

$$\tilde{\Pi}_{\phi_1} U \left[\prod_{k=1}^{(d-1)/2} \Pi_{\phi_{2k}} U^{\dagger} \tilde{\Pi}_{\phi_{2k+1}} U \right]$$

Main difference between QSVT & eigenvalue transformation is that the Bloch sphere transformations of U switch between bases $\{|v_k\rangle\}_k$ & $\{|w_k\rangle\}_k$

How to block encode?

Many applications assume that
A is available ~~as~~ encoded as
a block in U.

To gain some intuition, suppose that
unitary A is available,
then controlled-A is a block
encoding of A.

- Specifically,

$$|0\rangle\langle 0| \otimes A + |1\rangle\langle 1| \otimes I$$

$$= \begin{bmatrix} A & 0 \\ 0 & I \end{bmatrix}$$, & so A is

available as block encoded operator.

If controlled A is not available, but $\exists$ eigenvector $|\lambda\rangle$ of A, can then do