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Lecture 17.1

- so far in the class, we have, in some cases, followed a paradigm of promoting an irreversible function to a reversible one, which is then used in q. computing.
- this is the case for Deutsch-Josza & Shor's algorithms & even Grover's makes use of a phase oracle.

- However, there is another way to ~~[illegible]~~ embed irreversibility, by looking at reduced dynamics of unitary evolution.

Consider a unitary defined as

$$U = \begin{bmatrix} A & \sqrt{I-A^2} \\ \sqrt{I-A^2} & -A \end{bmatrix}$$

where A is a square matrix such that $\|A\|_\infty \leq 1$, then U is unitary & can be written as

$$\sigma_z \otimes A + \sigma_x \otimes \sqrt{I-A^2} .$$

Its action on the state

$$|\psi\rangle = \sqrt{p}\,|0\rangle|\psi_0\rangle + \sqrt{1-p_1}\,|1\rangle|\psi_1\rangle$$

is

$$U|\psi\rangle = \sqrt{p}\,|0\rangle A|\psi_0\rangle + \sqrt{1-p}\,|0\rangle \sqrt{I-A^2}|\psi_1\rangle$$
$$+ \sqrt{p}\,|1\rangle \sqrt{I-A^2}|\psi_0\rangle \quad \sqrt{1-p}\,|1\rangle \quad |\psi_1\rangle$$

Thus if we prepare

$|0\rangle|\psi_0\rangle$ then this goes to

$$|0\rangle A|\psi_0\rangle + |1\rangle \sqrt{I-A^2}\,|\psi_0\rangle$$

& measuring gives the outcome $|0\rangle$

w/ probability $\| A|\psi_0\rangle \|_2^2$

& post-meas. state is

$$\frac{A|\psi_0\rangle}{\|A|\psi_0\rangle\|_2}.$$

In this way, we can realize non-unitary irreversible dynamics.

This is a key aspect of QSVT (q. singular value transformation)

What QSVT allows for is performing a polynomial function $P(\cdot)$ of degree d using $O(d)$ elementary unitaries & realizing

$$\frac{P(A)|\psi_0\rangle}{\|P(A)|\psi_0\rangle\|_2}$$

this is so powerful that one can recover many known q. algorithms in this framework

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Let us begin w/ q. signal processing manipulating the first entry in a 2×2 unitary.

Use 2 kinds of single-qubit rotations:

1) a signal unitary $W(a)$:

$$W(a) = \begin{bmatrix} a & i\sqrt{1-a^2} \\ i\sqrt{1-a^2} & a \end{bmatrix}$$

(another convention is $R(a) = \begin{bmatrix} a & \sqrt{1-a^2} \\ \sqrt{1-a^2} & -a \end{bmatrix}$

this is an x-rotation by an angle $\theta = -2\cos^{-1}(a)$

i.e. $W(a) = \exp(-iX\theta/2)$

2) Signal processing unitary is

$$S(\phi) = e^{i\phi Z}$$

which is a Z-rotation by an angle -2ϕ, i.e.,

$$S(\phi) = e^{-iZ(-2\phi)/2}$$

Can then define a <u>signal processing</u> sequence as

$$U_{\vec{\phi}} = e^{i\phi_0 Z} \prod_{k=1}^{d} W(a)\, e^{i\phi_k Z}$$

where $\vec{\phi} = (\phi_0, \phi_1, \ldots, \phi_d) \in \mathbb{R}^{d+1}$

Idea is to use a signal processing sequence to modify top-left entry

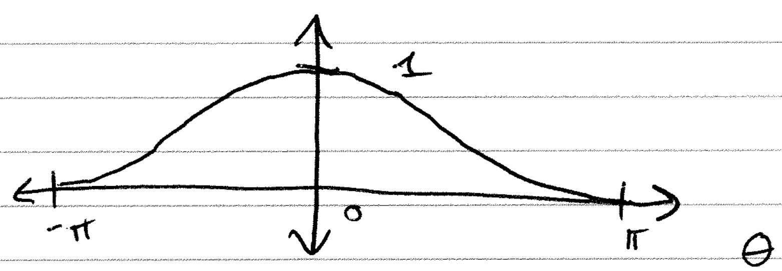
Let us consider some examples

Pick $\vec{\Phi} = (0,0)$. Implies no processing & then

$U_{\vec{\Phi}} = W(a)$ (unchanged signal)

can then plot $|\langle 0|U_{\vec{\Phi}}|0\rangle|^2 = |a|^2$

$= \cos^2(\theta/2)$ as



Now do some signal processing.

Pick $\vec{\Phi} = (\pi/2, -\eta, 2\eta, 0, -2\eta, \eta)$

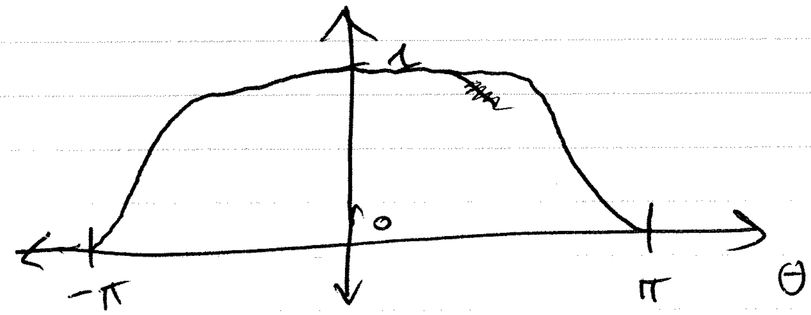
where $\eta = \frac{1}{2}\cos^{-1}(-1/4)$

Then $p = |\langle 0|U_{\vec{\Phi}}|0\rangle|^2$

$$= \frac{1}{8}\cos^2\left(\frac{\theta}{2}\right)\left[3\cos^8(\theta/2) - 15\cos^6\left(\frac{\theta}{2}\right) + 35\cos^4(\theta/2) - 45\cos^2(\theta/2) + 30\right]$$

$\approx 1 - \frac{5}{8}\left(\frac{\theta}{2}\right)^6$ for small θ

Now plot looks like



featuring a sharp transition @ $\theta \approx 2\pi/3$

Thus realizes a threshold function, a transformation of the original signal
– this is a famous pulse sequence in NMR called the BB1 sequence.

In general, the matrix element $P(a) = \langle 0 | U_{\vec{\phi}} | 0 \rangle$ is a polynomial in a, w/ degree no more than d.

Other examples:

$\vec{\phi} = (0,0) \Rightarrow P(a) = a$

$\vec{\phi} = (0,0,0) \Rightarrow p(a) = 2a^2 - 1$

$\vec{\phi} = (0,0,0,0) \Rightarrow P(a) = 4a^3 - 3a$

Chebyshev polynomials of the 1st kind

the reverse of these statements is possible

For a given polynomial $P(a)$ satisfying some constraints, $\exists$ a set $\vec{\phi}$ of signal processing phase angles such that

$P(a) = \langle 0 | U_{\vec{\phi}} | 0 \rangle$

More specifically,

Given complex polynomials P & Q

satisfying

1) $\deg(P) \leq d$, $\deg(Q) \leq d-1$

2) $\text{parity}(P) = d \bmod 2$ (even or odd function)

$\text{parity}(Q) = (d-1) \bmod 2$

3) $\cancel{|P|^2} \quad |P(x)|^2 + (1-x^2)\,|Q(x)|^2 = 1$

$\forall x \in [-1, 1]$

then $\exists\, \vec{\Phi} = (\phi_0, \phi_1, \ldots, \phi_d)$ such that

$$e^{i\phi_0 Z} \prod_{k=1}^{d} W(a)\, e^{i\phi_k Z}$$

$$= \begin{bmatrix} P(a) & iQ(a)\sqrt{1-a^2} \\ iQ^*(a)\sqrt{1-a^2} & P^*(a) \end{bmatrix}$$

- In practice, not interested in unitaries but rather in poly transformations of input a

- If we pick $\mathrm{Poly}(a) = \langle 0 | U_{\vec{\phi}} | 0 \rangle$ then transformations are limited.

- If we instead pick

$$\mathrm{Poly}(a) = \langle + | U_{\vec{\phi}} | + \rangle$$

this is equal to

$$\langle + | U_{\vec{\phi}} | + \rangle = RE\{P(a)\} + i\, RE\{Q(a)\}\sqrt{1-a^2}$$

- can then be shown that we can accurately approximate any real polynomial $R(x)$ such that

$\mathrm{Parity}(R) = d \bmod 2$,

$\deg(R) \leq d$, + $|R(a)| \leq 1 \quad \forall a \in [-1, 1]$

Much more general this way.

- Basis employed for polynomial is called signal basis & we take it to be $\{|+\rangle, |-\rangle\}$.

- can find phases using Remez-type exchange algorithms

Let us apply this to amplitude amplification & unstructured search

Main idea: reduce a multiqubit problem to a qubit subspace, a process called <u>qubitization</u>

Suppose we have access to a unitary U & two unitaries

$$A_\phi = e^{i\phi |A_0\rangle\langle A_0|}$$

$$B_\phi = e^{i\phi |B_0\rangle\langle B_0|}$$

(suppose we can realize any phase)

each of which rotates phases of specific states $|A_0\rangle$ & $|B_0\rangle$

GOAL: construct a circuit Q using $U, U^\dagger, A_\phi,$ & B_ϕ such that

$$|\langle A_0 | Q | B_0\rangle| \to 1$$

under assumption that $|\langle A_0 | U | B_0\rangle| > 0$.

This is a generalization of amplitude amplification. How?

$|B_0\rangle = |0\rangle$, $U = \mathcal{A}$ (algorithm),

$|A_0\rangle = |w\rangle$ (solution or marked element)

Note that we can solve this problem w/o knowledge of $\langle A_0|U|B_0\rangle$ by employing oblivious fixed-point amplitude amplification.

To begin, note that $U|B_0\rangle$ has a non-zero component along $|A_0\rangle$ & another component perpendicular to $|A_0\rangle$, defined as

$$|A_\perp\rangle = \frac{1}{N}\left(I - |A_0\rangle\langle A_0|\right) U |B_0\rangle$$

Then

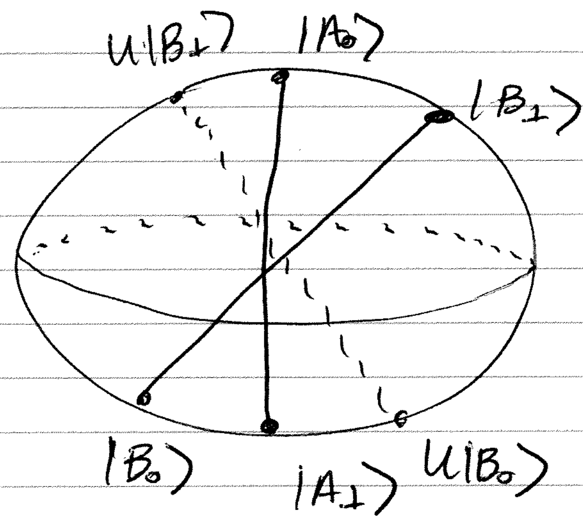
$$U|B_0\rangle = a|A_0\rangle + \sqrt{1-a^2}\,|A_\perp\rangle$$

(assume $a = \langle A_0|U|B_0\rangle$ is real by absorbing phase into $|B_0\rangle$)

can define $|B_\perp\rangle$ such that

$$\mathcal{U}|B_\perp\rangle = -a|A_\perp\rangle + \sqrt{1-a^2}\,|A_0\rangle$$

can write these qubit subspaces in terms of Bloch spheres



action of $\mathcal{U}$ is then a 2×2 unitary

$$\mathcal{U} = a\left(|A_0\rangle\langle B_0| - |A_\perp\rangle\langle B_\perp|\right) + \sqrt{1-a^2}\left(|A_\perp\rangle\langle B_0| + |A_0\rangle\langle B_\perp|\right)$$

$$\mathcal{U} = \begin{array}{c c} & \begin{array}{cc} |B_0\rangle & |B_\perp\rangle \end{array} \\ \begin{array}{c} \langle A_0| \\ \langle A_\perp| \end{array} & \begin{bmatrix} a & \sqrt{1-a^2} \\ \sqrt{1-a^2} & -a \end{bmatrix} \end{array} =: R(a)$$

can employ q. signal processing to establish the following:

Given access to U, $U^\dagger$, A_ϕ, B_ϕ,
& given polynomial $P(a)$, s.t. $\deg(P) \le d$
there exists a phase vector $\vec{\phi}$ s.t.

$$\langle A_0 | A_{\phi_0} \left[\prod_{k=1}^{\frac{d-1}{2}} U B_{\phi_{2k-1}} U^\dagger A_{\phi_{2k}} \right] U | B_0 \rangle = P(a)$$

~~Where~~

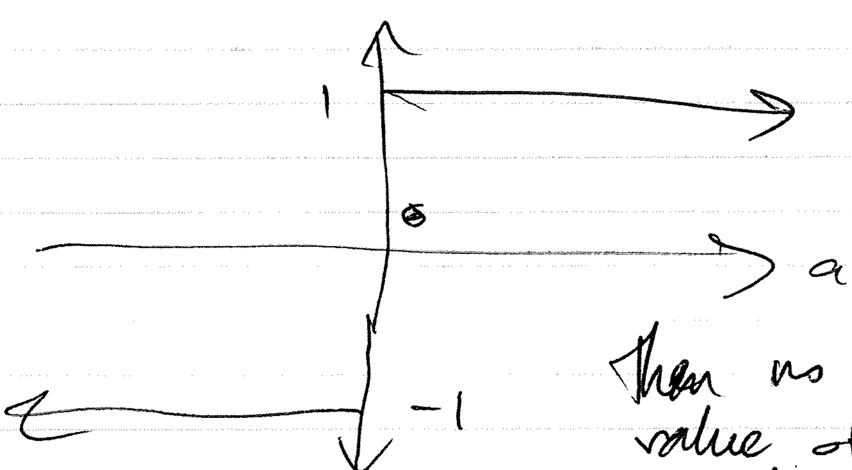
Proof: on subspaces defined by
$\mathrm{span}\{|A_0\rangle, |A_\perp\rangle\}$ &
$\mathrm{span}\{|B_0\rangle, |B_\perp\rangle\}$,
we have that $U = U^\dagger$.
Also, two-dim. rep. of U is $R(a) = -i\, e^{i\frac{\pi}{4}Z}\, W(a)\, e^{i\frac{\pi}{4}Z}$
Substituting above & noting that
A_ϕ & B_ϕ become z-axis rotations

$\Rightarrow$ desired equality reduces to

$$\langle A_0 | e^{i\phi_0' Z} \left[\prod_{k=1}^{d} W(a) e^{i\phi_k' Z} \right] U | B_0 \rangle$$
$$= P(a)$$

This is then a case of previous theorem.

For oblivious amplitude amplification, pick polynomial $P(a)$ to approximate the sign function



then no matter the value of a, it gets mapped to $\underline{1}$.

Grover's alg. then corresponds to

$$|A_0\rangle = |w\rangle,$$

$$A_\pi = e^{i\pi |A_0\rangle\langle A_0|},$$

$$U = H^{\otimes n},$$

$$|B_0\rangle = |0\rangle$$

$$\Rightarrow \langle A_0|U|B_0\rangle = 2^{-n/2} = \sqrt{1/N}$$