Lecture 24

Now let us discuss a specific entanglement measure called squashed entanglement.

To motivate it, consider the following measure of a bipartite state ρ_{AB}:

$$\inf_{\tau_A, \sigma_B} D(\rho_{AB} \| \tau_A \otimes \sigma_B) = I(A;B)_\rho$$

equal to minimum rel. entr. between ρ_{AB} + the set of product states. Equal to zero ^minimum value of iff a state is product.

It is thus a ~~classical~~ correlation measure + it is non-zero for ~~some~~ separable states that are not product states.

Recall that an entanglement should be equal to its minimum value on separable states.

Also, it should not increase under LOCC.

So mutual information ~~on~~ its own is not good for an Ent. measure.

However, we can still use mutual information in a different way.

Suppose that σ_{AB} is separable, i.e.,

$$\sigma_{AB} = \sum_x p(x)\, \rho_A^x \otimes \tau_B^x$$

This state has an extension of the form

$$\omega_{ABX} = \sum_x p(x)\, \rho_A^x \otimes \tau_B^x \otimes |x\rangle\langle x|_X$$

Then consider that

$$I(A;B|X)_\omega = \sum_x p(x)\, I(A;B)_{\rho^x \otimes \tau^x}$$
$$= 0$$

So this is a nice property that exploits the Markov structure of a separable state

$$\begin{tikzcd} & A \\ X \arrow[ur] \arrow[dr] & \\ & B \end{tikzcd}$$

(i.e. A & B are independent when X is given)

One potential measure of entanglement is thus

$$\inf_{\omega_{ABX}} \left\{ I(A;B|X)_\omega : \mathrm{Tr}_X[\omega_{ABX}] = \rho_{AB} \right\}$$

optimization is over extensions ω_{ABX} of the form

$$\omega_{ABX} = \sum_x p(x)\, \rho^x_{AB} \otimes |x\rangle\langle x|_X$$

satisfying $\rho_{AB} = \sum_x p(x) \rho_{AB}^x$.

quantity ≥ 0 $\forall$ states

& $= 0$ $\forall$ separable states

Idea for squashed entanglement is to take this one step further.

- Just extend w/ a general q. system rather than a classical system.
- Squashed entanglement is defined as

$$E_{sq}(A;B)_\rho = \frac{1}{2} \inf_{\omega_{ABE}} \left\{ I(A;B|E)_\omega : \mathrm{Tr}_E[\omega_{ABE}] = \rho_{AB} \right\}.$$

not known whether infimum can be replaced w/ a minimum.

squashed entanglement possesses many desirable properties

1) non-negativity $E_{sq}(A;B)_\rho \geq 0$

$\forall \rho_{AB} \in \mathcal{D}(\mathcal{H}_{AB})$

direct consequence of strong subadditivity $I(A;B|E) \geq 0$

2) faithfulness: $E_{sq}(A;B) = 0$

iff ρ_{AB} is separable

suppose ρ_{AB} separable.

Then an extension is

$$\sum_x p(x)\, \tau_A^x \otimes \omega_B^x \otimes |x\rangle\langle x|_E$$

for which $I(A;B|E) = 0$

matches lower bound & so

$E_{sq}(A;B)_\rho = 0 \quad \forall \rho_{AB} \in SEP$

other implication beyond scope.

3. Convexity:

$$E_{sq}(A;B)_{\bar{\rho}} \leq \sum_x p(x) E_{sq}(A;B)_{\rho^x}$$

Proof: Let ω^x_{ABE} denote an arbitrary extension of ρ^x_{AB}. Then

$$\omega_{ABEX} = \sum_x p(x) \omega^x_{ABE} \otimes |x\rangle\langle x|_X$$

extends $\bar{\rho}_{AB}$

$$\Rightarrow 2 \cdot E_{sq}(A;B)_{\bar{\rho}} \leq I(A;B|EX)_\omega$$

$$= \sum_x p(x) I(A;B|E)_{\omega^x}$$

Since this holds for arbitrary extensions, we conclude convexity.

4. squashed entanglement is additive:

For every state $\rho_{A_1 A_2 B_1 B_2}$

$$(*) \quad E_{sq}(A_1 A_2 ; B_1 B_2)_\rho \geq E_{sq}(A_1 ; B_1)_\rho + E_{sq}(A_2 ; B_2)_\rho$$

For a tensor-product state,

$$\omega_{A_1 B_1} \otimes \tau_{A_2 B_2} = \sigma_{A_1 A_2 B_1 B_2}$$

$$(**) \quad E_{sq}(A_1 A_2 ; B_1 B_2)_\sigma = E_{sq}(A_1 ; B_1)_\omega + E_{sq}(A_2 ; B_2)_\tau$$

1st prove $(*)$:

Let $\omega_{A_1 A_2 B_1 B_2 E}$ extend $\rho_{A_1 A_2 B_1 B_2}$

Then

$$I(A_1 A_2 ; B_1 B_2 | E)_\omega = I(A_1 ; B_1 B_2 | E)_\omega + I(A_2 ; B_1 B_2 | E A_1)_\omega$$

$$= I(A_1;B_1|E) + I(A_1;B_2|EB_1)$$

$$+ I(A_2;B_2|EA_1) + I(A_2;B_1|EA_1B_2)$$

$$\geq I(A_1;B_1|E) + I(A_2;B_2|EA_1)$$

$$\geq 2\left[E_{sq}(A_1;B_1)_\rho + E_{sq}(A_2;B_2)_\rho\right]$$

Since ω is an arbitrary extension, conclusion follows.

For additivity on product states, let $\omega_{A_1B_1E_1}$ extend $\omega_{A_1B_1}$ & let $\tau_{A_2B_2E_2}$ extend $\tau_{A_2B_2}$.

Then $\omega_{A_1B_1E_1} \otimes \tau_{A_2B_2E_2}$ extends $\omega_{A_1B_1} \otimes \tau_{A_2B_2}$

$$2 \cdot E_{sq}(A_1A_2;B_1B_2)_\sigma \leq I(A_1A_2;B_1B_2|E_1E_2)_{\omega\otimes\tau}$$

$$= I(A_1;B_1|E_1)_\omega + I(A_2;B_2|E_2)_\tau$$

<u>thm:</u> squashed ent. is a selective LOCC monotone.

<u>Proof:</u> 1) show it does not increase under local channels

2) show it is invariant under classical comm.

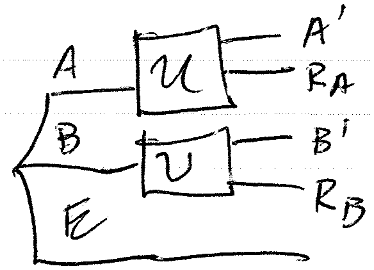
1) consider that conditional mutual information does not increase under local channels, i.e,

$$I(A;B|E)_\rho \geq I(A';B'|E)_\rho$$

where $\rho_{A'B'E} = (\mathcal{N}_{A\to A'} \otimes \mathcal{M}_{B\to B'})(\rho_{ABE})$

this follows b/c each channel has an isometric extension ~~and~~

$\mathcal{U}_{A\to A'R_A}$ + $\mathcal{V}_{B\to B'R_B}$



$$\& \; I(A;B|E) = I(A'R_A; B'R_B|E)$$

$$= I(A'; B'R_B|E) + I(R_A; B'R_B|EA')$$

$$= I(A'; B'|E) + I(A'; R_B|EB')$$

$$+ I(R_A; B'R_B|EA')$$

$$\geq I(A'; B'|E)$$

then data-processing for CMI

& optimizing gives

$$E_{sq}(A;B)_\rho \geq E_{sq}(A';B')_{(\mathcal{N}\otimes\mathcal{M})(\rho)}$$

Invariance under classical comm.

Let $\rho_{XAB} = \sum_x p(x) |x\rangle\langle x| \otimes \rho_{AB}^x$.

GOAL: prove that

$$E_{sq}(XA;B)_\rho = E_{sq}(A;BX)_\rho$$
$$= \sum_x p(x) E_{sq}(A;B)_{\rho^x}$$

Proof: Appending or discarding $|x\rangle\langle x|_X$ is a local channel, so that

$$E_{sq}(A;B)_{\rho^x} = E_{sq}(XA;B)_{|x\rangle\langle x| \otimes \rho^x}$$

From convexity, we get that

$$\sum_x p(x) E_{sq}(A;B)_{\rho^x} \geq E_{sq}(XA;B)_\rho$$

Similarly,

$$\sum_x p(x) E_{sq}(A;B)_{\rho^x} \geq E_{sq}(A;BX)_\rho$$

Now ~~consider that~~ we should prove the opposite inequalities.

An arbitrary extension ρ_{XABE} of ρ_{XAB} has the form

$$\rho_{XABE} = \sum_x p(x)\, |x\rangle\langle x|_X \otimes \rho^x_{ABE}$$

Then

$$2 \sum_x p(x)\, E_{sq}(A;B)_{\rho^x}$$

$$\leq \sum_x p(x)\, I(A;B|E)_{\rho^x}$$

$$= I(A;B|EX)_\rho$$

$$\leq I(XA;B|E)_\rho$$

Inequality holds for an arbitrary ext, so we conclude

$$\sum_x p(x)\, E_{sq}(A;B)_{\rho^x} \leq E_{sq}(XA;B)_\rho$$

Similar argument gives that

$$\sum_x p(x)\, E_{sq}(A;B)_{\rho^x} \leq E_{sq}(A;BX)_\rho$$

$\Rightarrow$

$$\sum_x p(x)\, E_{sq}(A;B)_{\rho^x} = E_{sq}(XA;B)_\rho$$
$$= E_{sq}(A;BX)_\rho$$

Squashed Entanglement for pure states

For ψ_{AB},

$$E_{sq}(A;B)_\psi = H(A)_\psi$$

An arbitrary extension of a pure state has the form $\psi_{AB} \otimes \omega_E$

$\Rightarrow \frac{1}{2} I(A;B|E) = \frac{1}{2} I(A;B)_\psi = H(A)_\psi$

one-shot distillable entanglement

$$E_d^{\varepsilon}(\rho_{AB}) = \sup_{\mathcal{L} \in LOCC} \left\{ \log_2 d : F(\mathcal{L}_{AB \to \hat{A}\hat{B}}(\rho_{AB}), \Phi^d_{\hat{A}\hat{B}}) \geq 1-\varepsilon \right\}$$

squashed entanglement upper bound

$$\begin{aligned} \log_2 d &= H(\hat{A})_{\Phi} \\ &= E_{sq}(\hat{A};\hat{B})_{\Phi} \quad \text{E_{sq} on pure states} \\ &\leq E_{sq}(\hat{A};\hat{B})_{\omega} + \sqrt{\varepsilon}\log_2 d + g_2(\sqrt{\varepsilon}) \quad \text{unif. cont. bound} \\ &\leq E_{sq}(A;B)_{\rho} + \text{''} \quad \text{LOCC monotone} \end{aligned}$$

$\Rightarrow$

$\cancel{E_d^{\varepsilon}(\rho_{AB}) \leq E_{sq}(A;B)_{\rho} + \sqrt{\varepsilon}\log_2}$

$$(1-\sqrt{\varepsilon})\log_2 d \leq E_{sq}(A;B)_{\rho} + g_2(\sqrt{\varepsilon})$$

$$\Rightarrow E_d^{\varepsilon}(\rho_{AB})$$
$$\leq \frac{1}{1-\sqrt{\varepsilon}}\left(E_{sq}(A;B)_\rho + g_2(\sqrt{\varepsilon})\right)$$

Asymptotic Distillable Entanglement

$$E_d(\rho_{AB}) = \inf_{\varepsilon\in(0,1)} \liminf_{n\to\infty} E_d^{\varepsilon}(\rho_{AB}^{\otimes n})$$

$$E_d(\rho_{AB}) \leq E_{sq}(A;B)_\rho \quad (*)$$

use one-shot bound

$$\frac{E_d^{\varepsilon}(\rho_{AB}^{\otimes n})}{n} \leq \frac{1}{1-\sqrt{\varepsilon}}\left(\frac{E_{sq}(A^n;B^n)_{\rho^{\otimes n}}}{n} + \frac{g_2(\sqrt{\varepsilon})}{n}\right)$$
$$= \frac{1}{1-\sqrt{\varepsilon}}\left(E_{sq}(A;B)_\rho + \frac{g_2(\sqrt{\varepsilon})}{n}\right) \quad \text{(additivity)}$$

(*) follows from taking limits.