Lecture 23

entanglement theory

Recall that a bipartite state is entangled if it cannot be written in the following separable form:

$$\sum_x p(x)\, \tau_A^x \otimes \omega_B^x$$

where $\{p(x)\}_x$ is a prob. dist. & $\{\tau_A^x\}_x$ & $\{\omega_B^x\}_x$ are sets of states.

A pure state ψ_{AB} is entangled iff its Schmidt rank > 1.

Also: If a bipartite state ρ_{AB} has negative partial transpose, then it is entangled.

- It is generally difficult to decide whether a state ρ_{AB} is entangled or not.

- The goal of entanglement theory is to quantify entanglement in an axiomatic or operational way.

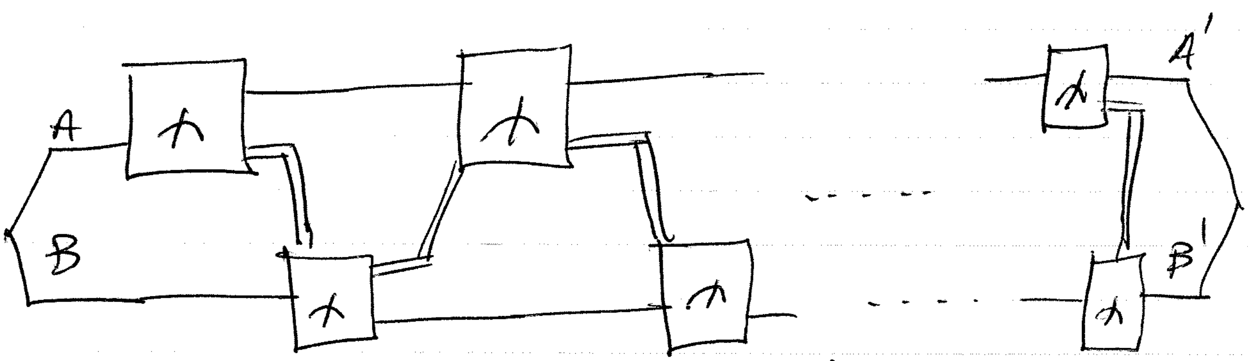
- What is the starting point for this theory?

Local operations alone should not increase entanglement, because entanglement is a kind of correlation & local operations do not increase correlations (even classical correlations)

Additionally, entanglement should not increase under classical comm. This should only increase the classical correlations, but not the entanglement.

Thus, entanglement should not increase under LOCC (i.e., local operations + classical comm.)

~~[illegible]~~ An LOCC channel has the form



composed of a finite # of one-way LOCC channels of the form:

→

1W-LOCC from Alice to Bob

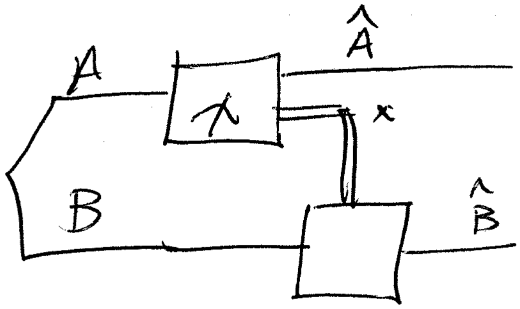
$$\rho_{AB} \rightarrow \sum_x \left(\mathcal{E}^x_{A\to\hat{A}} \otimes \tilde{\mathcal{F}}^x_{B\to\hat{B}}\right)(\rho_{AB})$$

where $\{\mathcal{E}^x_{A\to\hat{A}}\}_x$ is a set of CP maps such that

$\sum_x \mathcal{E}^x_{A\to\hat{A}}$ is trace preserving (q. instrument)

& $\{\tilde{\mathcal{F}}^x_{B\to\hat{B}}\}_x$ is a set of q. channels.

Picture for this is



1W-LOCC from Bob to Alice is defined as above, but roles of Alice & Bob swapped.

- Note that an LOCC channel $\mathcal{L}_{AB\to A'B'}$ can be written as a separable channel

$$\mathcal{L}_{AB\to A'B'}(\rho_{AB}) = \sum_z (\mathcal{M}^z_{A\to A'} \otimes \mathcal{N}^z_{B\to B'})(\rho_{AB})$$

where $\{\mathcal{M}^z\}_z$ & $\{\mathcal{N}^z\}_z$ are CP maps such that the sum map $\sum_z \mathcal{M}^z \otimes \mathcal{N}^z$ is trace preserving.

- However, there exist examples of separable channels that cannot be written as LOCC channels

Basic axiom of entanglement theory

$E(A;B)_\rho$ is an <u>entanglement measure</u>
if it is non-increasing under the
action of LOCC; i.e., if

$$E(A;B)_\rho \geq E(A';B')_\omega ,$$

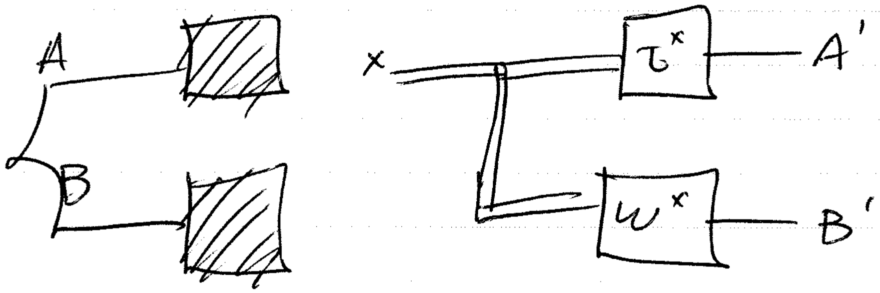
where $\omega_{A'B'} = \mathcal{L}_{AB \to A'B'}(\rho_{AB})$,

holds for every bipartite ~~state~~ state
ρ_{AB} + every LOCC channel $\mathcal{L}$.

this is related to data-processing
but is <u>a restricted</u> kind

This axiom implies that an ent. measure takes its minimum value on separable states. Why?

- can get from an arbitrary state ρ_{AB} to an arbitrary separable state $\sigma_{A'B'}$ by means of LOCC. Alice & Bob each trace out their systems locally & use classical comm. to generate $\sigma_{A'B'}$



this is LOCC & so

$$E(A;B)_\rho \geq E(A';B')_\sigma$$

$$\Rightarrow \quad E(A';B')_\sigma \geq E(A'';B'')_{\sigma'}$$

where $\sigma'_{A''B''}$ is separable,

$$\Rightarrow \quad E(A'';B'')_{\sigma'} \geq E(A';B')_\sigma$$

$$\Rightarrow \quad E(A';B')_\sigma = c \quad \forall \text{ separable states } \sigma_{A'B'}$$

if $c \neq 0$,
might as well ~~to~~ subtract c
such that

$$E(A';B')_\sigma = 0 \quad \text{for every separable state } \sigma_{A'B'}$$

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so all of this implies

1) $E(A;B)_\rho \geq 0$ for every state ρ_{AB}

2) $E(A;B)_\sigma = 0$ if σ_{AB} is separable.

Other desirable properties for an entanglement measure:

i) Faithfulness:

$$E(A;B)_\sigma = 0 \text{ iff } \sigma_{AB} \text{ is separable}$$

$$\Rightarrow E(A;B)_\rho > 0 \text{ iff } \rho_{AB} \text{ is entangled.}$$

2) Invariance under classical comm.:

$$E(XA;B)_\rho = E(A;BX)_\rho$$

$$= \sum_x p(x)\, E(A;B)_{\rho^x}$$

for $\rho_{XAB} = \sum_x p(x)\, |x\rangle\langle x| \otimes \rho^x_{AB}$

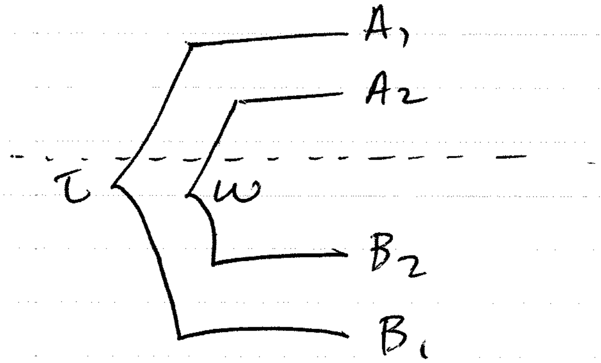
3) Convexity:

$$\sum_x p(x)\, E(\rho^x_{AB}) \geq E\left(\sum_x p(x)\, \rho^x_{AB}\right)$$

4) Additivity:

$$E(A_1A_2;B_1B_2)_{\tau\otimes\omega} = E(A_1;B_1)_\tau + E(A_2;B_2)_\omega$$

for the state $\tau_{A_1B_1} \otimes \omega_{A_2B_2}$



5. <u>Selective</u> LOCC monotonicity

Let $\{\mathcal{L}^x_{AB\to A'B'}\}_x$ be a collection of CP maps, such that the sum map

$$\sum_x \mathcal{L}^x_{AB\to A'B'}$$ is an LOCC channel

Each map $\mathcal{L}^x_{AB\to A'B'}$ is a composition of trace non-increasing one-way LOCC maps + can be written in the separable form

$$\mathcal{L}^{x}_{AB\to A'B'} = \sum_{y} \mathcal{E}^{x,y}_{A\to A'} \otimes \mathcal{F}^{x,y}_{B\to B'}$$

Ent. measure E satisfies selective LOCC monotonicity if

$$E(\rho_{AB}) \geq \sum_{x\,:\,p(x)\neq 0} p(x)\, E(\omega^{x}_{AB})$$

where $p(x) = \mathrm{Tr}[\mathcal{L}^{x}_{AB\to A'B'}(\rho_{AB})]$

& $\omega^{x}_{AB} = \dfrac{\mathcal{L}^{x}_{AB\to A'B'}(\rho_{AB})}{p(x)}$

Lemma: Suppose that E is a function obeying

1. data-processing under local channels
2. invariance under classical comm.

Then E is convex & is a selective LOCC monotone.

First prove convexity

Let $\rho_{XAB} = \sum_x p(x) |x\rangle\langle x|_X \otimes \rho^x_{AB}$

then $\sum_x p(x) E(A;B)_{\rho^x}$

$= E(XA;B)_\rho$ (invariance under classical comm.)

$\geq E(A;B)_\rho$ (discarding X system, local channel)

state in last line is

$\mathrm{Tr}_X[\rho_{XAB}] = \sum_x p(x) \rho^x_{AB}$.

To establish selective LOCC monotonicity, look at one-way LOCC channel of the form

$$\sum_{k,l} \mathcal{E}^{k,l}_{A\to A'} \otimes \mathcal{F}^{k,l}_{B\to B'}$$

where $\{\mathcal{E}^{k,l}\}_{k,l}$ is a set of CP maps s.t. $\sum_{k,l} \mathcal{E}^{k,l}$ is trace preserving

& $\{\mathcal{F}^{k,l}\}_{k,l}$ is a collection of q. channels

Use a superindex $m \equiv (k,l)$

think of k as classical info. that is kept, & l as being lost,

can implement 1W-LOCC channel in three steps

1. local Alice channel

$$\tau_{AB} \to \sum_m \mathcal{E}^m_{A\to A'}(\tau_{AB}) \otimes |m\rangle\langle m|_{M_A}$$

2. classical comm.

$$(\cdot)_{M_A} \to \sum_m |m\rangle_{M_B} \langle m|_{M_A} (\cdot) |m\rangle_{M_A} \langle m|_{M_B}$$

3. local Bob channel

$$(\cdot)_{BM_B} \to \sum_m \mathcal{F}^m_{B\to B'}(\cdot) \otimes |m\rangle\langle m|_{M_B} (\cdot) |m\rangle\langle m|_{M_B}$$

After 3,
global state becomes

$$\sum_m \left(\mathcal{E}^m_{A\to A'} \otimes \tilde{\mathcal{F}}^m_{B\to B'}\right)(\tau_{AB}) \otimes |m\rangle\langle m|_{M_B}$$

4. Bob discards l part so that
overall state becomes

$$= \sum_{k,l}\left(\mathcal{E}^{k,l} \otimes \tilde{\mathcal{F}}^{k,l}\right)(\tau_{AB}) \otimes |k,l\rangle\langle k,l|$$

$$\to \sum_{k,l}\left(\mathcal{E}^{k,l} \otimes \tilde{\mathcal{F}}^{k,l}\right)(\tau_{AB}) \otimes |k\rangle\langle k|$$

$$= \sum_k p(k)\, \omega^k_{A'B'} \otimes |k\rangle\langle k|$$

where $p(k) = \mathrm{Tr}\left[\sum_l \left(\mathcal{E}^{k,l} \otimes \tilde{\mathcal{F}}^{k,l}\right)(\tau_{AB})\right]$

$$\omega^k_{A'B'} = \frac{1}{p(k)} \sum_l \left(\mathcal{E}^{k,l} \otimes \tilde{\mathcal{F}}^{k,l}\right)(\tau_{AB})$$

We can now track how the entanglement changes

$$
\begin{aligned}
& E(A;B)_\tau \\
\geq\ & E(A' M_A ; B) && \text{(data-proc.-local channel)} \\
=\ & E(A' ; B M_B) && \text{(invariance under classical comm.)} \\
\geq\ & E(A' ; B' M_B) && \text{(data-proc. local channel)} \\
=\ & E(A' ; B' K_B L_B) && \\
\geq\ & E(A' ; B' K_B) && \\
=\ & \sum_k p(k)\, E(A' ; B')_{\omega^k} && \text{(invariance under classical comm.)}
\end{aligned}
$$

this argument was for one round of one-way LOCC, but we can keep applying recursively for each step.