Lecture 21

Continuing w/ properties of Petz-Renyi relative entropy

Isometric invariance

$$D_\alpha(\rho\|\sigma) = D_\alpha(V\rho V^\dagger \| V\sigma V^\dagger)$$

for V an isometry

To get this, let us 1st establish a formula for Petz-Renyi. Consider spectral decompositions of ρ + σ as

$$\rho = \sum_x p(x) |\psi^x\rangle\langle\psi^x|$$

$$\sigma = \sum_y q(y) |\varphi^y\rangle\langle\varphi^y|$$

$$\Rightarrow \operatorname{Tr}[\rho^\alpha \sigma^{1-\alpha}] = \operatorname{Tr}\Big[\Big(\sum_x p(x) |\psi^x\rangle\langle\psi^x|\Big)^\alpha \cdot \Big(\sum_y q(y)|\varphi^y\rangle\langle\varphi^y|\Big)^{1-\alpha}\Big]$$

$$= \mathrm{Tr}\left[\left(\sum_x p(x)^\alpha |\psi^x\rangle\langle\psi^x|\right)\left(\sum_y q(y)^{1-\alpha} |\varphi^y\rangle\langle\varphi^y|\right)\right]$$

$$= \sum_{x,y} p(x)^\alpha q(y)^{1-\alpha} \mathrm{Tr}\left[|\psi^x\rangle\langle\psi^x|\varphi^y\rangle\langle\varphi^y|\right]$$

$$= \sum_{x,y} p(x)^\alpha q(y)^{1-\alpha} |\langle\psi^x|\varphi^y\rangle|^2$$

$\Rightarrow$ isometric invariance b/c

$$D_\alpha(V\rho V^\dagger \| V\sigma V^\dagger)$$

$$= \frac{1}{\alpha-1}\log \sum_{x,y} p(x)^\alpha q(y)^{1-\alpha} |\langle\psi^x|\underbrace{V^\dagger V}_{I}|\varphi^y\rangle|^2$$

Now to prove data-processing inequality

$$D_\alpha(\rho\|\sigma) \geq D_\alpha(\mathcal{N}(\rho)\|\mathcal{N}(\sigma))$$

for ρ a state, σ PSD, $\forall \alpha \in (0,1) \cup (1,2)$

Since we have isometric invariance, it suffices to prove that

$$D_\alpha(\rho_{AB} \| \sigma_{AB}) \geq D_\alpha(\rho_A \| \sigma_A)$$

(data processing under partial trace)

prove that

$$Q_\alpha(\rho_{AB} \| \sigma_{AB}) \geq Q_\alpha(\rho_A \| \sigma_A) \text{ for } \alpha \in (1,2]$$

&

$$Q_\alpha(\rho_{AB} \| \sigma_{AB}) \leq Q_\alpha(\rho_A \| \sigma_A) \text{ for } \alpha \in (0,1)$$

We can write

$$Q_\alpha(\rho_{AB} \| \sigma_{AB}) = \langle \varphi^{\rho_{AB}} | f(\rho_{AB}^{-1} \otimes \sigma_{\hat{A}\hat{B}}^{T}) | \varphi^{\rho_{AB}} \rangle$$

$$Q_\alpha(\rho_A \| \sigma_A) = \langle \varphi^{\rho_A} | f(\rho_A^{-1} \otimes \sigma_{\hat{A}}^{T}) | \varphi^{\rho_A} \rangle$$

for $f(x) = x^{1-\alpha}$

& $|\varphi^{\rho_{AB}}\rangle = \left(\sqrt{\rho_{AB}} \otimes I_{\hat{A}\hat{B}}\right) |\Gamma\rangle_{A\hat{A}} \otimes |\Gamma\rangle_{B\hat{B}}$

$|\varphi^{\rho_A}\rangle = \left(\sqrt{\rho_A} \otimes I_{\hat{A}}\right)|\Gamma\rangle_{A\hat{A}}$

canonical purifications of ρ_{AB} & ρ_A

Define the isometry

$$V_{A\hat{A} \to AB\hat{A}\hat{B}} = \sqrt{\rho_{AB}} \left(\sqrt{\rho_A^{-1}} \otimes I_{\hat{A}}\right)|\Gamma\rangle_{B\hat{B}}$$

$\Rightarrow V_{A\hat{A} \to AB\hat{A}\hat{B}} |\psi\rangle_{A\hat{A}}$

$$= \sqrt{\rho_{AB}} \left(\sqrt{\rho_A^{-1}} \otimes I_{\hat{A}}\right) |\psi\rangle_{A\hat{A}} |\Gamma\rangle_{B\hat{B}}$$

Isometry because

$$V^\dagger V = \langle\Gamma|_{B\hat{B}} \left(\sqrt{\rho_A^{-1}} \otimes I_{\hat{A}}\right) \sqrt{\rho_{AB}} \sqrt{\rho_{AB}} \cdot \left(\sqrt{\rho_A^{-1}} \otimes I_{\hat{A}}\right) |\Gamma\rangle_{B\hat{B}}$$

$$= \langle\Gamma|_{B\hat{B}} \sqrt{\rho_A^{-1}}\, \rho_{AB} \sqrt{\rho_A^{-1}}\, |\Gamma\rangle_{B\hat{B}}$$

(5)

$$= \sqrt{\rho_A^{-1}}\, \langle\Gamma|_{B\hat{B}}\, \rho_{AB} |\Gamma\rangle_{B\hat{B}} \sqrt{\rho_A^{-1}}$$

$$= I_{\hat{A}} \otimes \sqrt{\rho_A^{-1}}\, \rho_A \sqrt{\rho_A^{-1}} = I_{A\hat{A}}$$

Also

$$V_{A\hat{A}\to AB\hat{A}\hat{B}} |\varphi^{\rho_A}\rangle$$

$$= \sqrt{\rho_{AB}} \left(\sqrt{\rho_A^{-1}} \otimes I_{\hat{A}}\right) \left(\sqrt{\rho_A} \otimes I_{\hat{A}}\right) |\Gamma\rangle_{A\hat{A}} |\Gamma\rangle_{B\hat{B}}$$

$$= \sqrt{\rho_{AB}}\, |\Gamma\rangle_{A\hat{A}} |\Gamma\rangle_{B\hat{B}} = |\varphi^{\rho_{AB}}\rangle_{AB\hat{A}\hat{B}}$$

Finally,

$$V^\dagger \left(\rho_{AB}^{-1} \otimes \sigma_{\hat{A}\hat{B}}^T\right) V$$

$$= \langle\Gamma|_{B\hat{B}}\, \rho_A^{-1/2} \rho_{AB}^{1/2} \left(\rho_{AB}^{-1} \otimes \sigma_{\hat{A}\hat{B}}^T\right) \rho_{AB}^{1/2} \rho_A^{-1/2} |\Gamma\rangle_{B\hat{B}}$$

$$= \langle\Gamma|_{B\hat{B}}\, \rho_A^{-1/2} \left(I_{AB} \otimes \sigma_{\hat{A}\hat{B}}^T\right) \rho_A^{-1/2} |\Gamma\rangle_{B\hat{B}}$$

$$= \rho_A^{-1/2} \langle\Gamma|_{B\hat{B}}\, I_{AB} \otimes \sigma_{\hat{A}\hat{B}}^T |\Gamma\rangle_{B\hat{B}}\, \rho_A^{-1/2}$$

$$= \rho_A^{-1/2} \left(I_A \otimes \sigma_{\hat{A}}^T\right) \rho_A^{-1/2} = \rho_A^{-1} \otimes \sigma_{\hat{A}}^T$$

After all this setup,
proof is one line!!

$$Q_\alpha(\rho_{AB}\|\sigma_{AB}) = \langle\varphi^{\rho_A}|V^\dagger f(\rho_{AB}^{-1}\otimes\sigma_{\hat{A}\hat{B}}^T)V|\varphi^{\rho_A}\rangle$$
$$\geq \langle\varphi^{\rho_A}|f(V^\dagger(\rho_{AB}^{-1}\otimes\sigma_{\hat{A}\hat{B}}^T)V)|\varphi^{\rho_A}\rangle$$
$$= \langle\varphi^{\rho_A}|f(\rho_A^{-1}\otimes\sigma_{\hat{A}}^T)|\varphi^{\rho_A}\rangle$$
$$= Q_\alpha(\rho_A\|\sigma_A)$$

operator Jensen inequality for $\alpha\in(1,2]$
for $\alpha\in(0,1)$, inequality is opposite direction

thus operator convexity/concavity
is at the heart
of q. data processing

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data-processing for q. relative entropy
follows from this & by applying
a limit.

Other properties of Petz-Renyi

Additivity:

$$D_\alpha(\rho_1\otimes\rho_2\|\sigma_1\otimes\sigma_2) = D_\alpha(\rho_1\|\sigma_1) + D_\alpha(\rho_2\|\sigma_2)$$

Direct-sum property:

For $\rho_{XA} = \sum_x p(x)\,|x\rangle\langle x|\otimes\rho^x$

$\sigma_{XA} = \sum_x q(x)\,|x\rangle\langle x|\otimes\sigma^x$

$$Q_\alpha(\rho_{XA}\|\sigma_{XA}) = \sum_x p(x)^\alpha q(x)^{1-\alpha}\, Q_\alpha(\rho^x\|\sigma^x)$$

This latter one & data processing imply joint convexity of Q_α for $\alpha\in(1,2]$ & joint concavity of Q_α for $\alpha\in(0,1)$