Lecture 20

generalized divergence

function $\underline{D}$ that obeys data processing

$$\underline{D}(\rho\|\sigma) \geq \underline{D}(\mathcal{N}(\rho)\|\mathcal{N}(\sigma))$$

for every state ρ, PSD σ, & channel $\mathcal{N}$.

$$\Rightarrow \quad \underline{D}(\rho\|\sigma) = \underline{D}(V\rho V^\dagger \| V\sigma V^\dagger)$$

for isometry V.

(same argument as rel. entr.)

$$\Rightarrow \quad \underline{D}(\rho\|\sigma) = \underline{D}(\rho\otimes\tau \| \sigma\otimes\tau)$$

for every state τ

follows b/c appending τ is a channel, as well as partial trace

$\Rightarrow D(\rho\|\rho) = c \quad \forall$ states ρ

follows b/c $D(\rho\|\rho) \geq D(\omega\|\omega)$
& vice versa for all states ρ & ω.
Data processing using trace & replace channel

then we can simply set $D(\rho\|\rho) = 0$
by subtracting the constant c.

Also, if $D(1\|c) \geq 0 \quad \forall c \in (0,1]$
then

$D(\rho\|\sigma) \geq 0 \quad \forall$ states ρ & σ PSD: $Tr[\sigma] \leq 1$

Petz - Renyi relative entropy

$$D_\alpha(\rho\|\sigma) = \frac{1}{\alpha-1} \log Q_\alpha(\rho\|\sigma)$$

$$Q_\alpha(\rho\|\sigma) = \mathrm{Tr}[\rho^\alpha \sigma^{1-\alpha}] \quad \forall \alpha \in (0,1) \cup (1,\infty)$$

need $\mathrm{supp}(\rho) \subseteq \mathrm{supp}(\sigma)$ for $\alpha \in (1,\infty)$ in order for $Q_\alpha(\rho\|\sigma)$ to be finite.

Renyi entropy recovered as

$$H_\alpha(\rho) = -D_\alpha(\rho\|I)$$

$$= \frac{1}{1-\alpha} \log_2 \mathrm{Tr}[\rho^\alpha] = \frac{\alpha}{1-\alpha} \log_2 \|\rho\|_\alpha$$

in limit $\alpha \to \{0, 1, \infty\}$

$$H_0(\rho) = \log_2 \mathrm{rank}(\rho)$$

$$H_1(\rho) = -\mathrm{Tr}[\rho \log_2 \rho]$$

$$H_\infty(\rho) = -\log \|\rho\|_\infty = -\log \lambda_{max}(\rho)$$

$$\lim_{\alpha \to 1} D_\alpha(\rho\|\sigma) = D(\rho\|\sigma)$$

Proof use natural log.

just focus on case $\mathrm{supp}(\rho) \subseteq \mathrm{supp}(\sigma)$ (*)

Define $Q_{\alpha,\beta}(\rho\|\sigma) = \mathrm{Tr}[\rho^\alpha \sigma^{1-\beta}]$

$\Rightarrow \quad Q_\alpha = Q_{\alpha,\alpha}$

(*) $\Rightarrow \quad Q_1(\rho\|\sigma) = \mathrm{Tr}[\rho \Pi_\sigma] = 1$

$$\Rightarrow D_\alpha(\rho\|\sigma) = \frac{\ln Q_\alpha(\rho\|\sigma) - \ln Q_1(\rho\|\sigma)}{\alpha - 1}$$

$$\Rightarrow \lim_{\alpha \to 1} D_\alpha(\rho\|\sigma) = \frac{d}{d\alpha} \ln Q_\alpha(\rho\|\sigma) \Big|_{\alpha=1}$$

$$= \frac{1}{Q_1(\rho\|\sigma)} \frac{d}{d\alpha} Q_\alpha(\rho\|\sigma) \Big|_{\alpha=1}$$

$$= \frac{d}{d\alpha} Q_\alpha(\rho\|\sigma) \Big|_{\alpha=1}$$

Using the function $Q_{\alpha,\beta}$ & the chain rule, we can write

$$\frac{d}{d\alpha} Q_\alpha(\rho\|\sigma)\Big|_{\alpha=1}$$

$$= \frac{d}{d\alpha} Q_{\alpha,1}(\rho\|\sigma)\Big|_{\alpha=1}^{(1)} + \frac{d}{d\beta} Q_{1,\beta}(\rho\|\sigma)\Big|_{\beta=1}^{(2)}$$

$$\frac{d}{d\alpha} Q_{\alpha,1}(\rho\|\sigma) = \frac{d}{d\alpha} \mathrm{Tr}[\rho^\alpha \Pi_\sigma]$$

$$= \frac{d}{d\alpha} \mathrm{Tr}[\rho^\alpha]$$

$$= \mathrm{Tr}[\rho^\alpha \ln \rho]$$

$$\Rightarrow (1) = \mathrm{Tr}[\rho \ln \rho]$$

$$\frac{d}{d\beta} Q_{1,\beta}(\rho\|\sigma) = \frac{d}{d\beta} \mathrm{Tr}\{\rho\, \sigma^{1-\beta}\}$$

$$= -\mathrm{Tr}[\rho\, \sigma^{1-\beta} \ln \sigma]$$

$$\Rightarrow (2) = -\mathrm{Tr}[\rho\, \Pi_\sigma \ln \sigma]$$

$$= -\mathrm{Tr}\{\rho \ln \sigma\}$$

$$\Rightarrow \lim_{\alpha \to 1} D_\alpha(\rho\|\sigma) = \mathrm{Tr}\{\rho(\ln \rho - \ln \sigma)\}$$

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can write Petz - Renyi in terms of

$$Q_\alpha(\rho\|\sigma) = \langle \varphi^\rho | (\rho^{-1} \otimes \sigma^T)^{1-\alpha} | \varphi^\rho \rangle$$

where $|\varphi^\rho\rangle = (\rho^{1/2} \otimes I)|\Gamma\rangle$

Follows because

$$\mathrm{Tr}[\rho^\alpha \sigma^{1-\alpha}] = \mathrm{Tr}[\rho^{1/2} \rho^{\alpha-1} \rho^{1/2} \sigma^{1-\alpha}]$$

$$= \langle \Gamma | \rho^{1/2} \rho^{\alpha-1} \rho^{1/2} \sigma^{1-\alpha} \otimes I | \Gamma \rangle$$

$$= \langle \Gamma | \rho^{1/2} \rho^{\alpha-1} \rho^{1/2} \otimes (\sigma^T)^{1-\alpha} | \Gamma \rangle$$

$$= \langle \Gamma | (\rho^{1/2} \otimes I)(\rho^{\alpha-1} \otimes (\sigma^T)^{1-\alpha})(\rho^{1/2} \otimes I) | \Gamma \rangle$$
$$= \langle \varphi^\rho | (\rho^{-1} \otimes \sigma^T)^{\alpha-1} | \varphi^\rho \rangle$$

can use this to prove monotonicity in α

i.e.,
$$D_\alpha(\rho\|\sigma) \leq D_\beta(\rho\|\sigma)$$

for $0 \leq \alpha \leq \beta$

Proof: write

$$D_\alpha(\rho\|\sigma) = \frac{1}{\alpha-1} \ln \langle \varphi^\rho | X^{1-\alpha} | \varphi^\rho \rangle$$

where $X = \rho^{-1} \otimes \sigma^T$

$$= -\frac{1}{\gamma} \ln \langle \varphi^\rho | X^\gamma | \varphi^\rho \rangle$$

where $\gamma = 1-\alpha$

Since $\frac{d}{d\alpha} = \frac{d}{d\gamma}\frac{d\gamma}{d\alpha} = \frac{-d}{d\gamma}$

$$\Rightarrow \frac{d}{d\alpha} D_\alpha(\rho\|\sigma) = \frac{d}{d\gamma} \frac{1}{\gamma} \ln \langle \varphi^\rho | X^\gamma | \varphi^\rho \rangle$$

$$= \frac{-1}{\gamma^2} \ln \langle \varphi \rho | X^\gamma | \varphi \rho \rangle + \frac{1}{\gamma} \frac{\langle \varphi \rho | X^\gamma \ln X | \varphi \rho \rangle}{\langle \varphi \rho | X^\gamma | \varphi \rho \rangle}$$

$$= \frac{-\langle \varphi \rho | X^\gamma | \varphi \rho \rangle \ln \langle \varphi \rho | X^\gamma | \varphi \rho \rangle + \gamma \langle \varphi \rho | X^\gamma \ln X | \varphi \rho \rangle}{\gamma^2 \langle \varphi \rho | X^\gamma | \varphi \rho \rangle}$$

$$= \frac{-\langle \varphi \rho | X^\gamma | \varphi \rho \rangle \ln \langle \varphi \rho | X^\gamma | \varphi \rho \rangle + \langle \varphi \rho | X^\gamma \ln X^\gamma | \varphi \rho \rangle}{\gamma^2 \langle \varphi \rho | X^\gamma | \varphi \rho \rangle}$$

Set $g(x) = x \ln x$

$$\Rightarrow \frac{d}{d\alpha} D_\alpha(\rho \| \sigma) = \frac{\langle \varphi \rho | g(X^\gamma) | \varphi \rho \rangle - g(\langle \varphi \rho | X^\gamma | \varphi \rho \rangle)}{\gamma^2 \langle \varphi \rho | X^\gamma | \varphi \rho \rangle}$$

g is operator convex $\Rightarrow$

operator Jensen $\Rightarrow$

$$\langle \varphi_\rho | g(X_\sigma) | \varphi_\rho \rangle \geq g(\langle \varphi_\rho | X_\sigma | \varphi_\rho \rangle)$$

$$\Rightarrow \frac{d}{d\alpha} D_\alpha(\rho \| \sigma) \geq 0$$

$\Rightarrow$ D_α monotonically increasing in α.