Lecture 18

Fidelity is an alternative measure, but it is a similarity measure.

Start with the definition for pure states $|\psi\rangle\langle\psi|$ & $|\phi\rangle\langle\phi|$

$$F(\psi, \phi) = |\langle\psi|\phi\rangle|^2$$

simply the squared overlap of the state vectors.

Interpretation: probability that $|\phi\rangle$ would pass a test for being $|\psi\rangle$.

Specifically, the test is the measurement $\{|\psi\rangle\langle\psi|, I - |\psi\rangle\langle\psi|\}$.

where the first outcome corresponds to "pass." Probability of "pass" is then

$$\mathrm{Tr}[|\psi\rangle\langle\psi|\,|\phi\rangle\langle\phi|] = |\langle\psi|\phi\rangle|^2$$
$$= F(\psi, \phi)$$

Observe that fidelity is symmetric in the states.

Generalizing this formula to a density operator & a pure state ψ, we define fidelity as

$$F(\psi, \rho) = \mathrm{Tr}[|\psi\rangle\langle\psi|\rho]$$
$$= \langle\psi|\rho|\psi\rangle$$

~~In~~ In this way, it retains the interpretation as the prob.

that ρ passes a test for being $|\psi\rangle\langle\psi|$.

How to generalize fidelity
to arbitrary density operators?

could consider

$$\mathrm{Tr}[\rho\sigma]$$

& argue that this is probability
that σ would pass a test
for being ρ, according
to measurement
$\{\rho, I-\rho\}$.

However, big problem w/ this def.:

fidelity should be equal to one
iff states are the same.

But not true for this formula:
pick $\rho=\sigma=I/d$.

Idea: go back to pure state formula & optimize over all purifications of ρ & σ:

$$F(\rho, \sigma) = \max_{|\psi^\rho\rangle_{RS}, |\psi^\sigma\rangle_{RS}} |\langle\psi^\rho|\psi^\sigma\rangle_{RS}|^2$$

where

$$\mathrm{Tr}_R[\psi^\rho_{RS}] = \rho_S \;\&$$

$$\mathrm{Tr}_R[\psi^\sigma_{RS}] = \sigma_S \,.$$

then fidelity $= 1$ if ρ & σ are the same.

Since all purifications are related by an isometry acting on ref. system we can optimize fidelity as

So fidelity is

$$F(\rho,\sigma) = \max_{\substack{V^{\rho}_{R\to R'},\\ V^{\sigma}_{R\to R'}}} \left| \langle \psi^{\rho}|_{RS} (V^{\rho}_{R\to R'} \otimes I_S)^{\dagger} (V^{\sigma}_{R\to R'} \otimes I_S) |\psi^{\sigma}\rangle_{RS} \right|^2$$

$$= \max_{''} \left| \langle \psi^{\rho}|_{RS} (V^{\rho\dagger} V^{\sigma})_R \otimes I_S |\psi^{\sigma}\rangle_{RS} \right|^2$$

can be shown

$$= \max_{U_R} \left| \langle \psi^{\rho}|_{RS} U_R \otimes I_S |\psi^{\sigma}\rangle_{RS} \right|^2$$

(alternatively, set V^{ρ} & V^{σ} to be unitaries)

This is called Uhlmann formula for fidelity.

Can show that

$$F(\rho,\sigma) = \|\sqrt{\rho}\sqrt{\sigma}\|_1^2$$

$$= \left(\mathrm{Tr}\left[\sqrt{\sqrt{\sigma}\,\rho\sqrt{\sigma}} \right] \right)^2$$

well known standard formula for fidelity.

Let's prove this. Consider that

$$\|A\|_1 = \max_{U} |\mathrm{Tr}[AU]|$$

where U is unitary.

Then

$$\|\sqrt{\rho}\sqrt{\sigma}\|_1 = \max_{U} |\mathrm{Tr}[\sqrt{\rho}\sqrt{\sigma}U]|$$

$$\cancel{= \max_{U} |\langle\Gamma|_{RS} (\sqrt{\rho}\sqrt{\sigma}U)_R \otimes I_S |\Gamma\rangle|}$$

$$= \max_{U} |\langle\Gamma|_{RS}\, I_R \otimes (\sqrt{\rho}\sqrt{\sigma}U)_S |\Gamma\rangle_{RS}|$$

$$= \max_{U} |\langle\Gamma|_{RS}\, U_R^T \otimes \sqrt{\rho}\sqrt{\sigma} |\Gamma\rangle_{RS}|$$

$$= \max_{U} |\langle\Gamma|_{RS} (I_R \otimes \sqrt{\rho_S})(U_R^T \otimes I_S)(I_R \otimes \sqrt{\sigma_S})|\Gamma\rangle_{RS}|$$

$$= \max_{U} |\langle\psi^\rho|_{RS}\, U_R^T \otimes I_S |\psi^\sigma\rangle_{RS}|$$

done...

Fidelity can be evaluated by a semi-definite program

$$\sqrt{F}(\rho,\sigma) = \frac{1}{2} \max_X \left\{ \mathrm{Tr}[X] + \mathrm{Tr}[X^{\dagger}] : \begin{pmatrix} \rho & X \\ X^{\dagger} & \sigma \end{pmatrix} \geq 0 \right\}$$

$$= \frac{1}{2} \min_{Y,Z \geq 0} \left\{ \mathrm{Tr}[Y\rho] + \mathrm{Tr}[Z\sigma] : \begin{bmatrix} Y & I \\ I & Z \end{bmatrix} \geq 0 \right\}$$

Properties of fidelity

1) $F(\rho,\sigma) \in [0,1]$

it is equal to zero iff states are orthogonal $\rho\sigma = 0$

+ equal to one iff states are identical.

Suppose that $\rho = \sigma$.

Then $\|\sqrt{\rho}\sqrt{\sigma}\|_1^2 = \|\sqrt{\rho}\sqrt{\rho}\|_1^2$

$= \|\rho\|_1^2 = 1$

Sps. that $\rho\sigma = 0$

then $\mathrm{Tr}\{\rho\sigma\} = 0$

$= \mathrm{Tr}[\sqrt{\sigma}\,\rho\sqrt{\sigma}] = 0$

Since $\sqrt{\sigma}\,\rho\sqrt{\sigma}$ is PSD

$\Rightarrow \sqrt{\sigma}\,\rho\sqrt{\sigma} = 0$

$\Rightarrow \quad \|\sqrt{\rho}\sqrt{\sigma}\|_1^2 = \left(\mathrm{Tr}\left[\sqrt{\sqrt{\sigma}\,\rho\sqrt{\sigma}}\right]\right)^2$

$= 0$.

Suppose that $F(\rho,\sigma) = 0$

then $\|\sqrt{\rho}\sqrt{\sigma}\|_1^2 = 0 \Rightarrow \sqrt{\rho}\sqrt{\sigma} = 0$

$\Rightarrow \rho\sigma = 0$

Suppose that $F(\rho,\sigma)=1$

By Uhlmann's theorem, $\exists$ exists unitary U_R such that

$$\left|\langle\psi^\rho|_{RS}\, U_R\otimes I_S\, |\psi^\sigma\rangle_{RS}\right|^2 = 1$$

unitary can be chosen such that

$$\langle\psi^\rho|\, U\otimes I\, |\psi^\sigma\rangle = 1$$

$$\Rightarrow \quad |\psi^\rho\rangle = U\otimes I\,|\psi^\sigma\rangle$$

$\Rightarrow$ reduced states are the same.

2) Fidelity invariant under isometries

$$F(\rho,\sigma) = F(V\rho V^\dagger, V\sigma V^\dagger)$$

follows because

$$\|\sqrt{V\rho V^\dagger}\sqrt{V\sigma V^\dagger}\|_1^2 = \|V\sqrt{\rho}V^\dagger V\sqrt{\sigma}V^\dagger\|_1^2$$
$$= \|V\sqrt{\rho}\sqrt{\sigma}V^\dagger\|_1^2$$

$$= \|\sqrt{\rho}\sqrt{\sigma}\|_1^2$$

3) Fidelity is multiplicative :

$$F(\rho_1 \otimes \rho_2, \sigma_1 \otimes \sigma_2) = F(\rho_1, \sigma_1) \cdot F(\rho_2, \sigma_2)$$

use definition...

Data-processing inequality

$$F(\rho, \sigma) \leq F(\mathcal{N}(\rho), \mathcal{N}(\sigma))$$

where $\mathcal{N}$ is a channel

Recall that $\mathcal{N}(\cdot) = \mathrm{Tr}_E[V(\cdot)V^\dagger]$

for an isometry V

Since $F(\rho, \sigma) = F(V\rho V^\dagger, V\sigma V^\dagger)$

it suffices to prove data-proc under partial trace.

Let ρ_{AB} & σ_{AB} be bipartite states

w/ purifications $|\psi^\rho\rangle_{RAB}$ &

$|\psi^\sigma\rangle_{RAB}$

then ψ^ρ_{RAB} purifies ρ_A &

ψ^σ_{RAB} " σ_A.

Using Uhlmann's theorem;

$$F(\rho_{AB}, \sigma_{AB}) = \max_{U_R} \left|\langle\psi^\rho|_{RAB} U_R \otimes I_{AB} |\psi^\sigma\rangle_{RAB}\right|^2$$

$$\leq \max_{U_{RB}} \left|\langle\psi^\rho|_{RAB} U_{RB} \otimes I_A |\psi^\sigma\rangle_{RAB}\right|^2$$

$$= F(\rho_A, \sigma_A).$$

Then proof for channels is

$$F(N(\rho), N(\sigma)) = F(\mathrm{Tr}_E[V\rho V^\dagger], \mathrm{Tr}_E[V\sigma V^\dagger])$$

$$\geq F(V\rho V^\dagger, V\sigma V^\dagger)$$

$$= F(\rho, \sigma)$$

Joint concavity of root fidelity

$$\sqrt{F}\left(\sum_x p(x)\rho^x, \sum_x p(x)\sigma^x\right) \geq \sum_x p(x)\sqrt{F}(\rho^x, \sigma^x)$$

<u>Proof:</u> pick cq states

$$\rho_{XA} = \sum_x p(x)\,|x\rangle\langle x|_X \otimes \rho_A^x$$

$$\sigma_{XA} = \sum_x p(x)\,|x\rangle\langle x| \otimes \sigma_A^x$$

then $F(\rho_{XA}, \sigma_{XA}) \leq F(\rho_A, \sigma_A)$

$$= F\left(\sum_x p(x)\rho_A^x, \sum_x p(x)\sigma_A^x\right)$$

Now evaluate

$$\sqrt{F}(\rho_{XA}, \sigma_{XA}) = \|\sqrt{\rho_{XA}}\sqrt{\sigma_{XA}}\|_1$$

$$= \Big\|\left(\sum_x |x\rangle\langle x|_X \otimes \sqrt{p(x)\rho^x}\right)\left(\sum_{x'} |x'\rangle\langle x'|_X \otimes \sqrt{p(x)\sigma^x}\right)\Big\|_1$$

$$= \left\| \sum_x |x\rangle\langle x|_X \otimes p(x) \sqrt{\rho^x} \sqrt{\sigma^x} \right\|_1$$

$$= \sum_x \| p(x) \sqrt{\rho^x} \sqrt{\sigma^x} \|_1$$

$$= \sum_x p(x) \| \sqrt{\rho^x} \sqrt{\sigma^x} \|_1$$

$$= \sum_x p(x) \sqrt{F}(\rho^x, \sigma^x)$$

Fidelity via measurement

$$F(\rho, \sigma) = \min_{\{\Lambda_x\}_x} \left(\sum_x \sqrt{\mathrm{Tr}[\Lambda_x \rho] \, \mathrm{Tr}[\Lambda_x \sigma]} \right)^2$$

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conclude "$\leq$" by taking $\mathcal{M}$
to be a measurement channel

$$\mathcal{M}(\omega) = \sum_x \mathrm{Tr}[\Lambda_x \omega] |x\rangle\langle x|$$

+ apply data processing

$$F(\rho, \sigma) \leq F(\mathcal{M}(\rho), \mathcal{M}(\sigma))$$

Now construct a POVM that achieves fidelity.

Set $A = \sigma^{-1/2}\left(\sigma^{1/2}\rho\,\sigma^{1/2}\right)^{1/2}\sigma^{-1/2}$

inverse on support.

and observe that

$F(\rho,\sigma) =$

$\left(\mathrm{Tr}\left[\sqrt{\sqrt{\sigma}\,\rho\sqrt{\sigma}}\right]\right)^2$

$= \left(\mathrm{Tr}[A\sigma]\right)^2$

Diagonalize A as $A = \sum_i r_i|\psi_i\rangle\langle\psi_i|$

Consider that $\mathrm{Tr}[A\sigma]$

$= \mathrm{Tr}\left[\sum_i r_i|\psi_i\rangle\langle\psi_i|\sigma\right]$

$= \sum_i r_i\langle\psi_i|\sigma|\psi_i\rangle$

$= \sum_i \sqrt{\langle\psi_i|r_i\sigma r_i|\psi_i\rangle}\sqrt{\langle\psi_i|\sigma|\psi_i\rangle}$

$$= \sum_i \sqrt{\langle \psi_i | A \sigma A | \psi_i \rangle} \sqrt{\langle \psi_i | \sigma | \psi_i \rangle}$$

$$= \sum_i \sqrt{\langle \psi_i | \rho | \psi_i \rangle} \sqrt{\langle \psi_i | \sigma | \psi_i \rangle}$$

$(A \sigma A = \rho)$

We showed that

$$F(\rho, \sigma) = \left(\sum_i \sqrt{\langle \psi_i | \rho | \psi_i \rangle \langle \psi_i | \sigma | \psi_i \rangle} \right)^2$$

so that measurement is

given by $\{ |\psi_i\rangle\langle\psi_i| \}_i$