Lecture 15

Q. Hypothesis testing

q. system prepared in state ρ or σ & objective is to guess which one was prepared by means of a q. measurement

$\{M_\rho, M_\sigma\}$ $\quad M_\rho, M_\sigma \geq 0$

$M_\rho + M_\sigma = I$

M_ρ: identified w/ guessing ρ

M_σ: identified w/ guessing σ

2 kinds of errors that can occur:

Type I: guess "σ", but ρ was prepared

probability of type I error $= \mathrm{Tr}[M_\sigma \rho]$

Type II error: guess "ρ", when prepared state is σ

Probability of Type II error $= \text{Tr}[M_\rho \sigma]$

Two different ways of combining these kinds of errors to arrive at a single notion of error:

Symmetric / Bayesian:

prior probability $\lambda \in [0,1]$ for being ρ & $1-\lambda$ for being σ

$$\Pr[\text{error}] = \Pr[\text{error} \mid \rho] \Pr[\rho] + \Pr[\text{error} \mid \sigma] \Pr[\sigma]$$

$$= \text{Tr}[(I - M)\rho]\lambda + \text{Tr}[M\sigma] \cdot (1-\lambda)$$

$$\equiv P_{err}(\lambda, \rho, \sigma, M)$$

goal is to minimize this error prob.

optimization problem is then

$$p_{err}^*(\lambda,\rho,\sigma) \equiv \min_M \{ p_{err}(\lambda,\rho,\sigma,M) : 0 \leq M \leq I \}$$

This is an SDP:

~~we~~ can write

$$p_{err}(\lambda,\rho,\sigma,M)$$
$$= \lambda + \mathrm{Tr}[M((1-\lambda)\sigma - \lambda\rho)]$$

$$\Rightarrow p_{err}^*(\lambda,\rho,\sigma) = \min_{M\geq 0} \{ \mathrm{Tr}[M((1-\lambda)\sigma - \lambda\rho)] : M \leq I \} + \lambda$$

other kind of hypothesis testing setting is

Asymmetric case: minimize Type II err. prob. subject to constraint on Type I error prob.:

$$\min_M \{ \mathrm{Tr}[M\sigma] : \mathrm{Tr}[(I-M)\rho] \leq \varepsilon, \; 0 \leq M \leq I \}$$

This is also an SDP!

Let us focus on the symmetric case 1st.

property of p_{err}^*:

obeys data processing ineq., i.e.,

$$p_{err}^*(\delta, \rho, \sigma) \leq p_{err}^*(\delta, \mathcal{N}(\rho), \mathcal{N}(\sigma))$$

Intuition: if you act w/ a channel 1st before doing an optimal measurement, then error cannot decrease.

Proof: let M' be an arbitrary measurement op. Then $\mathcal{N}^\dagger(M')$ is also a measurement op.

$$\Rightarrow \quad \lambda + \mathrm{Tr}\{M'((1-\lambda)\mathcal{N}(\sigma) - \lambda\mathcal{N}(\rho))\}$$

$$= \lambda + \mathrm{Tr}\{\mathcal{N}^{\dagger}(M')((1-\lambda)\sigma - \lambda\rho)\}$$

$$\geq \lambda + \min_{0 \leq M \leq I} \mathrm{Tr}[M((1-\lambda)\sigma - \lambda\rho)]$$

$$= p_{err}^{*}(\lambda, \rho, \sigma)$$

Since ineq. holds for all measurement op's, conclude that

$$p_{err}^{*}(\lambda, \mathcal{N}(\rho), \mathcal{N}(\sigma)) \geq p_{err}^{*}(\lambda, \rho, \sigma)$$

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closed-form expression for $p_{err}^{*}(\lambda, \rho, \sigma)$ via Helstrom - Holevo theorem:

$$\inf_{M} \left\{ \mathrm{Tr}[(I-M)A] + \mathrm{Tr}[MB] : 0 \leq M \leq I \right\} = \frac{1}{2}\left(\mathrm{Tr}[A+B] - \|A-B\|_1\right)$$

Let M be an arbitrary measurement op.

Set $\Delta = A - B$ & let Δ_+ & Δ_- be positive & negative parts of Δ. ($A - B = \Delta_+ - \Delta_-$ & $\Delta_+ \Delta_- = 0$)

can write

$$\mathrm{Tr}[(I-M)A] + \mathrm{Tr}[MB]$$
$$= \mathrm{Tr}[A] - (\mathrm{Tr}[M\Delta_+] - \mathrm{Tr}[M\Delta_-])$$

since $\mathrm{Tr}[M\Delta_-] \geq 0$,

follows that

$$\mathrm{Tr}[M\Delta_+] - \mathrm{Tr}[M\Delta_-] \leq \mathrm{Tr}[M\Delta_+]$$
$$\leq \mathrm{Tr}[\Delta_+]$$

$\Rightarrow$

$$\mathrm{Tr}[(I-M)A] + \mathrm{Tr}[MB] \geq \mathrm{Tr}[A] - \mathrm{Tr}[\Delta_+]$$

Since $\|A-B\|_1 = \mathrm{Tr}[|A-B|]$

$= \mathrm{Tr}[\Delta_+] + \mathrm{Tr}[\Delta_-]$ &

$\Delta_- = \Delta_+ + B - A$

$$\Rightarrow \mathrm{Tr}[\Delta_+] = \frac{1}{2}\left(\|A-B\|_1 - \mathrm{Tr}[B-A]\right)$$

$$\Rightarrow \inf_{\substack{M: \\ 0 \le M \le I}} \mathrm{Tr}[(I-M)A] + \mathrm{Tr}[MB] \geq \frac{1}{2}\left(\mathrm{Tr}[A+B] - \|A-B\|_1\right)$$

To see reverse inequality,

pick $M = \Pi_+$ (projection onto positive part of $A-B$)

then $\mathrm{Tr}[M\Delta_+] = \mathrm{Tr}[\Delta_+]$ +

$\mathrm{Tr}[M\Delta_-] = 0$

then we get equality

pick $A = \frac{1}{2}\rho$ $B = \frac{1}{2}\sigma$

$\Rightarrow$ $p^*_{err}(\frac{1}{2}, \rho, \sigma)$

$$= \frac{1}{2}\left(1 - \frac{1}{2}\|\rho - \sigma\|_1\right)$$

gives operational meaning to trace distance.