PHYS 7347: Quantum Information Theory	Fall 2021

Lecture 14

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1 Overview

In this lecture we discuss a striking difference between classical correlations and entanglement, as witnessed in the CHSH game (also known as *Bell's theorem*).

2 Entanglement in the CHSH Game

One of the simplest means for demonstrating the power of entanglement is with a two-player game known as the CHSH game (after Clauser, Horne, Shimony, and Holt), which is a particular variation of the original setup in Bell's theorem. We first present the rules of the game, and then we find an upper bound on the probability that players operating according to a classical strategy can win. We finally leave it as an exercise to show that players sharing a maximally entangled Bell state $|\Phi^+\rangle$ can have an approximately 10% higher chance of winning the game with a quantum strategy. This result, known as *Bell's theorem*, represents one of the most striking separations between classical and quantum physics.

The players of the game are Alice and Bob, who are spatially separated from each other from the time that the game starts until it is over. The game begins with a referee selecting two bits x and y uniformly at random. The referee then sends x to Alice and y to Bob. Alice and Bob are not allowed to communicate with each other in any way at this point. Alice sends back to the referee a bit a, and Bob sends back a bit b. Since they are spatially separated, Alice's response bit a cannot depend on Bob's input bit y, and similarly, Bob's response bit b cannot depend on Alice's input bit x. After receiving the response bits a and b, the referee determines if the AND of x and y is equal to the exclusive OR of a and b. If so, then Alice and Bob win the game. That is, the winning condition is

$$x \wedge y = a \oplus b. \tag{1}$$

Figure 1 depicts the CHSH game.

We need to figure out an expression for the winning probability of the CHSH game. Let $V(x, y, a, b)$ denote the following indicator function for whether they win in a particular instance of the game:

$$V(x, y, a, b) = \begin{cases} 1 & \text{if } x \wedge y = a \oplus b \\ 0 & \text{else} \end{cases}. \tag{2}$$

There is a conditional probability distribution $p_{AB|XY}(a, b|x, y)$, which corresponds to the particular strategy that Alice and Bob employ. Since the inputs x and y are chosen uniformly at random and

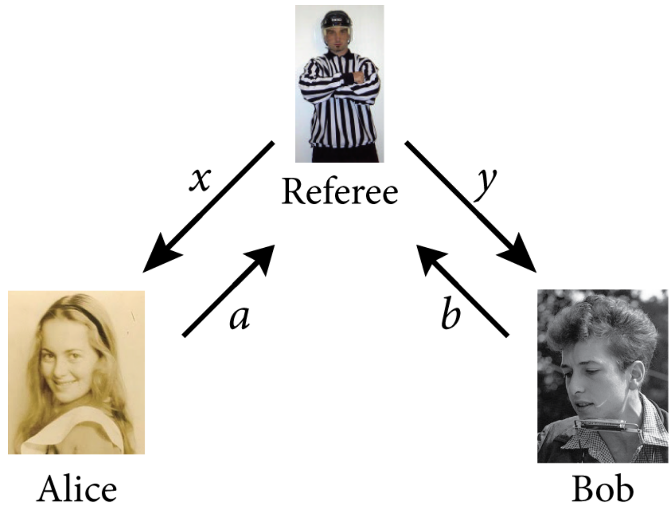

Figure 1: A depiction of the CHSH game. The referee distributes the bits x and y to Alice and Bob in the first round. In the second round, Alice and Bob return the bits a and b to the referee.

each take on two possible values, the distribution $p_{XY}(x, y)$ for x and y is as follows:

$$p_{XY}(x, y) = 1/4. \tag{3}$$

So an expression for the winning probability of the CHSH game is

$$\frac{1}{4} \sum_{a,b,x,y} V(x, y, a, b) p_{AB|XY}(a, b|x, y). \tag{4}$$

In order to calculate this winning probability for a classical or quantum strategy, we need to understand the distribution $p_{AB|XY}(a, b|x, y)$ further. In order to do so, we need a way for describing the strategy that Alice and Bob employ. For this purpose, we will assume that there is a random variable Λ taking values λ, which describes either a classical or quantum strategy, and its values could be all of the entries in a matrix and even taking on continuous values. Using the law of total probability, we can expand the conditional probability $p_{AB|XY}(a, b|x, y)$ as follows:

$$p_{AB|XY}(a, b|x, y) = \int d\lambda \; p_{AB|\Lambda XY}(a, b|\lambda, x, y) \; p_{\Lambda|XY}(\lambda|x, y), \tag{5}$$

where $p_{\Lambda|XY}(\lambda|x, y)$ is a conditional probability distribution. Decomposing the distribution $p_{AB|XY}(a, b|x, y)$ in this way leads to the depiction of their strategy given in Figure 2(i).

2.0.1 Classical Strategies

Let us suppose that they act according to a classical strategy. What is the most general form of such a strategy? Looking at the picture in Figure 2(i), there are a few aspects of it which are not consistent with our understanding of how the game works.

In a classical strategy, the random variable Λ corresponds to classical correlations that Alice and Bob can share *before the game begins*. They could meet beforehand and select a value λ of Λ at

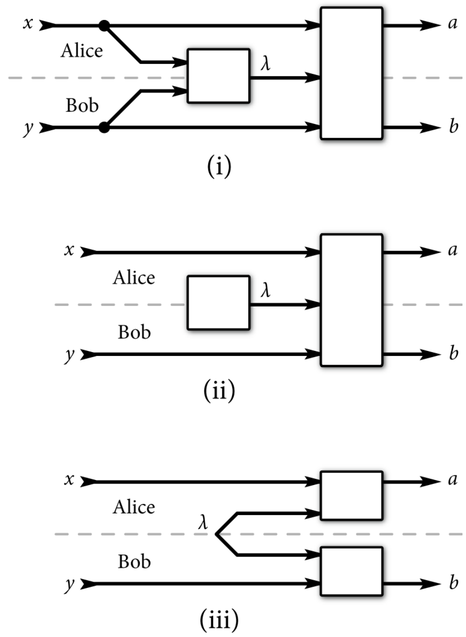

Figure 2: Various reductions of a classical strategy in the CHSH game: (i) an unconstrained strategy, (ii) strategy resulting from demanding that the parameter λ is independent of the input bits x and y, and (iii) further demanding that Alice and Bob's actions are independent and that they do not have access to each other's input bits.

random. According to the specification of the game, the input bits x and y for Alice and Bob are chosen independently at random, and so the random variable Λ cannot depend on the bits x and y. So the conditional distribution $p_{\Lambda|XY}(\lambda|x,y)$ simplifies as follows:

$$p_{\Lambda|XY}(\lambda|x,y) = p_\Lambda(\lambda), \tag{6}$$

and Figure 2(ii) reflects this constraint.

Next, Alice and Bob are spatially separated and acting independently, so that the distribution $p_{AB|\Lambda XY}(a,b|\lambda,x,y)$ factors as follows:

$$p_{AB|\Lambda XY}(a,b|\lambda,x,y) = p_{A|\Lambda XY}(a|\lambda,x,y)\; p_{B|\Lambda XY}(b|\lambda,x,y). \tag{7}$$

But we also said that Alice's strategy cannot depend on Bob's input bit y and neither can Bob's strategy depend on Alice's input x, because they are spatially separated. However, their strategies could depend on the random variable Λ, which they are allowed to share before the game begins. All of this implies that the conditional distribution describing their strategy should factor as follows:

$$p_{AB|\Lambda XY}(a,b|\lambda,x,y) = p_{A|\Lambda X}(a|\lambda,x)\; p_{B|\Lambda Y}(b|\lambda,y), \tag{8}$$

and Figure 2(iii) reflects this change. Now Figure 2(iii) depicts the most general classical strategy that Alice and Bob could employ if Λ corresponds to a random variable that Alice and Bob are both allowed to access before the game begins.

Putting everything together, the conditional distribution $p_{AB|XY}(a,b|x,y)$ for a classical strategy takes the following form:

$$p_{AB|XY}(a,b|x,y) = \int d\lambda \; p_{A|\Lambda X}(a|\lambda,x) \; p_{B|\Lambda Y}(b|\lambda,y) \; p_\Lambda(\lambda), \tag{9}$$

and we can now consider optimizing the winning probability in (4) with respect to all classical strategies. Consider that any stochastic map $p_{A|\Lambda X}(a|\lambda,x)$ can be simulated by applying a deterministic binary-valued function $f(a|\lambda,x,n)$ to a local random variable N taking values labeled by n. That is, we can always find a random variable N such that

$$p_{A|\Lambda X}(a|\lambda,x) = \int dn \; f(a|\lambda,x,n) \; p_N(n). \tag{10}$$

The same is true for the stochastic map $p_{B|\Lambda Y}(b|\lambda,y)$; i.e., there is a random variable M such that

$$p_{B|\Lambda Y}(b|\lambda,y) = \int dm \; g(b|\lambda,y,m) \; p_M(m), \tag{11}$$

where g is a deterministic binary-valued function. So this implies that

$$\begin{aligned}
&p_{AB|XY}(a,b|x,y) \\
&= \int d\lambda \; p_{A|\Lambda X}(a|\lambda,x) \; p_{B|\Lambda Y}(b|\lambda,y) \; p_\Lambda(\lambda) && (12)\\
&= \int d\lambda \left[\int dn \; f(a|\lambda,x,n) \; p_N(n)\right] \left[\int dm \; g(b|\lambda,y,m) \; p_M(m)\right] \; p_\Lambda(\lambda) && (13)\\
&= \int\int\int d\lambda \; dn \; dm \; f(a|\lambda,x,n) \;\; g(b|\lambda,y,m) \;\; p_\Lambda(\lambda) \; p_N(n) \; p_M(m). && (14)
\end{aligned}$$

By inspecting the last line above, it is clear that we could then have the shared random variable Λ subsume the local random variables N and M, allowing us to write any conditional distribution $p_{AB|XY}(a,b|x,y)$ for a classical strategy as follows:

$$p_{AB|XY}(a,b|x,y) = \int d\lambda \; f'(a|\lambda,x) \; g'(b|\lambda,y) \; p_\Lambda(\lambda), \tag{15}$$

where f' and g' are deterministic binary-valued functions (related to f and g). Substituting this expression into the winning probability expression in (4), we find that

$$\begin{aligned}
&\frac{1}{4}\sum_{a,b,x,y} V(x,y,a,b) p_{AB|XY}(a,b|x,y) \\
&= \frac{1}{4}\sum_{a,b,x,y} V(x,y,a,b) \int d\lambda \; f'(a|\lambda,x) \; g'(b|\lambda,y) \; p_\Lambda(\lambda) && (16)\\
&= \int d\lambda \; p_\Lambda(\lambda) \left[\frac{1}{4}\sum_{a,b,x,y} V(x,y,a,b) \; f'(a|\lambda,x) \; g'(b|\lambda,y)\right] && (17)\\
&\leq \frac{1}{4}\sum_{a,b,x,y} V(x,y,a,b) \; f'(a|\lambda^*,x) \; g'(b|\lambda^*,y). && (18)
\end{aligned}$$

In the second equality, we just exchanged the integral over λ with the sum. In the inequality in the last step, we used the fact that the average is always less than the maximum. That is, there is always a particular value λ^* that leads to a higher winning probability than when averaging over all values of λ. As a consequence of the above development, we see that it suffices to consider deterministic strategies of Alice and Bob when analyzing the winning probability.

Since we now know that deterministic strategies are optimal among all classical strategies, let us focus on these. A deterministic strategy would have Alice select a bit a_x conditioned on the bit x that she receives, and similarly, Bob would select a bit b_y conditioned on y. The following table presents the winning conditions for the four different values of x and y with this deterministic strategy:

x	y	$x \wedge y$	$= a_x \oplus b_y$
0	0	0	$= a_0 \oplus b_0$
0	1	0	$= a_0 \oplus b_1$
1	0	0	$= a_1 \oplus b_0$
1	1	1	$= a_1 \oplus b_1$

. (19)

However, we can observe that it is impossible for them to always win. If we add the entries in the column $x \wedge y$, the binary sum is equal to one, while if we add the entries in the column $= a_x \oplus b_y$, the binary sum is equal to zero. Thus, it is impossible for all of these equations to be satisfied. At most, only three out of four of them can be satisfied, so that the maximal winning probability with a classical deterministic strategy $p_{AB|XY}(a, b|x, y)$ is at most 3/4:

$$\frac{1}{4} \sum_{a,b,x,y} V(x, y, a, b) p_{AB|XY}(a, b|x, y) \leq \frac{3}{4}. \tag{20}$$

We can then see that a strategy for them to achieve this upper bound is for Alice and Bob always to return $a = 0$ and $b = 0$ no matter the values of x and y.

2.0.2 Quantum Strategies

What does a quantum strategy of Alice and Bob look like? Here the parameter λ can correspond to a shared quantum state $|\phi\rangle_{AB}$. Alice and Bob perform local measurements depending on the value of the inputs x and y that they receive. We can write Alice's x-dependent measurement as $\{\Pi_a^{(x)}\}$ where for each x, $\Pi_a^{(x)}$ is a projector and $\sum_a \Pi_a^{(x)} = I$. Similarly, we can write Bob's y-dependent measurement as $\{\Pi_b^{(y)}\}$. Then we instead employ the Born rule to determine the conditional probability distribution $p_{AB|XY}(a, b|x, y)$:

$$p_{AB|XY}(a, b|x, y) = \langle\phi|_{AB} \Pi_a^{(x)} \otimes \Pi_b^{(y)} |\phi\rangle_{AB}, \tag{21}$$

so that the winning probability with a particular quantum strategy is as follows:

$$\frac{1}{4} \sum_{a,b,x,y} V(x, y, a, b) \langle\phi|_{AB} \Pi_a^{(x)} \otimes \Pi_b^{(y)} |\phi\rangle_{AB}. \tag{22}$$

Interestingly, if Alice and Bob share a maximally entangled state, they can achieve a higher winning probability than if they share classical correlations only. This is one demonstration of the power of entanglement, and we leave it as an exercise to prove that the following quantum strategy achieves a winning probability of $\cos^2(\pi/8) \approx 0.85$ in the CHSH game.

Exercise 1. *Suppose that Alice and Bob share a maximally entangled state $|\Phi^+\rangle$. Show that the following strategy has a winning probability of $\cos^2(\pi/8)$. If Alice receives $x = 0$ from the referee, then she performs a measurement of Pauli Z on her system and returns the outcome as a. If she receives $x = 1$, then she performs a measurement of Pauli X and returns the outcome as a. If Bob receives $y = 0$ from the referee, then he performs a measurement of $(X + Z)/\sqrt{2}$ on his system and returns the outcome as b. If Bob receives $y = 1$ from the referee, then he performs a measurement of $(Z - X)/\sqrt{2}$ and returns the outcome as b.*

2.0.3 Maximum Quantum Winning Probability

Given that classical strategies cannot win with probability any larger than 3/4, it is natural to wonder if there is a bound on the winning probability of a quantum strategy. It turns out that $\cos^2(\pi/8)$ is the maximum probability with which Alice and Bob can win the CHSH game using a quantum strategy, a result known as *Tsirelson's bound.* To establish this result, let us go back to the CHSH game. Conditioned on the inputs x and y being equal to 00, 01, or 10, we know that Alice and Bob win if they report back the same results. The probability for this to happen with a given quantum strategy is

$$\langle\phi|_{AB}\, \Pi_0^{(x)} \otimes \Pi_0^{(y)} \, |\phi\rangle_{AB} + \langle\phi|_{AB}\, \Pi_1^{(x)} \otimes \Pi_1^{(y)} \, |\phi\rangle_{AB}\,, \tag{23}$$

and the probability for it not to happen is

$$\langle\phi|_{AB}\, \Pi_0^{(x)} \otimes \Pi_1^{(y)} \, |\phi\rangle_{AB} + \langle\phi|_{AB}\, \Pi_1^{(x)} \otimes \Pi_0^{(y)} \, |\phi\rangle_{AB}\,. \tag{24}$$

So, conditioned on x and y being equal to 00, 01, or 10, the probability of winning minus the probability of losing is

$$\langle\phi|_{AB}\, A^{(x)} \otimes B^{(y)} \, |\phi\rangle_{AB}\,, \tag{25}$$

where we define the *observables* $A^{(x)}$ and $B^{(y)}$ as follows:

$$A^{(x)} \equiv \Pi_0^{(x)} - \Pi_1^{(x)}, \tag{26}$$

$$B^{(y)} \equiv \Pi_0^{(y)} - \Pi_1^{(y)}. \tag{27}$$

If x and y are both equal to one, then Alice and Bob should report back different results, and similar to the above, one can work out that the probability of winning minus the probability of losing is equal to

$$-\langle\phi|_{AB}\, A^{(1)} \otimes B^{(1)} \, |\phi\rangle_{AB}\,. \tag{28}$$

Thus, when averaging over all values of the input bits, the probability of winning minus the probability of losing is equal to

$$\frac{1}{4}\langle\phi|_{AB}\, C_{AB} \, |\phi\rangle_{AB}\,, \tag{29}$$

where C_{AB} is the CHSH operator, defined as

$$C_{AB} \equiv A^{(0)} \otimes B^{(0)} + A^{(0)} \otimes B^{(1)} + A^{(1)} \otimes B^{(0)} - A^{(1)} \otimes B^{(1)}. \tag{30}$$

It is a simple exercise to check that

$$C_{AB}^2 = 4I_{AB} - \left[A^{(0)}, A^{(1)}\right] \otimes \left[B^{(0)}, B^{(1)}\right]. \tag{31}$$

The infinity norm $\|R\|_\infty$ of an operator R is equal to its largest singular value. It obeys the following relations:

$$\|cR\|_\infty = |c| \, \|R\|_\infty \,, \tag{32}$$
$$\|RS\|_\infty \leq \|R\|_\infty \, \|S\|_\infty \,, \tag{33}$$
$$\|R+S\|_\infty \leq \|R\|_\infty + \|S\|_\infty \,, \tag{34}$$

where $c \in \mathbb{C}$ and S is another operator. Using these, we find that

$$\begin{aligned}
\left\|C_{AB}^2\right\|_\infty &= \left\|4I_{AB} - \left[A^{(0)}, A^{(1)}\right] \otimes \left[B^{(0)}, B^{(1)}\right]\right\|_\infty &(35)\\
&\leq 4\,\|I_{AB}\|_\infty + \left\|\left[A^{(0)}, A^{(1)}\right] \otimes \left[B^{(0)}, B^{(1)}\right]\right\|_\infty &(36)\\
&= 4 + \left\|\left[A^{(0)}, A^{(1)}\right]\right\|_\infty \left\|\left[B^{(0)}, B^{(1)}\right]\right\|_\infty &(37)\\
&\leq 4 + 2 \cdot 2 = 8, &(38)
\end{aligned}$$

implying that

$$\|C_{AB}\|_\infty \leq \sqrt{8} = 2\sqrt{2}. \tag{39}$$

Given this and the expression in (29), the probability of winning minus the probability of losing can never be larger than $\sqrt{2}/2$ for any quantum strategy. Combined with the fact that the probability of winning summed with the probability of losing is equal to one, we find that the winning probability of any quantum strategy can never be larger than $1/2 + \sqrt{2}/4 = \cos^2(\pi/8)$.