Lecture 12

Recall Kraus representation of a q. channel:

$$\mathcal{N}_{A\to B}(X) = \sum_i K_i X K_i^\dagger$$

just as states & measurements can be purified, so can channels.

Stinespring theorem:

Every channel can be realized by

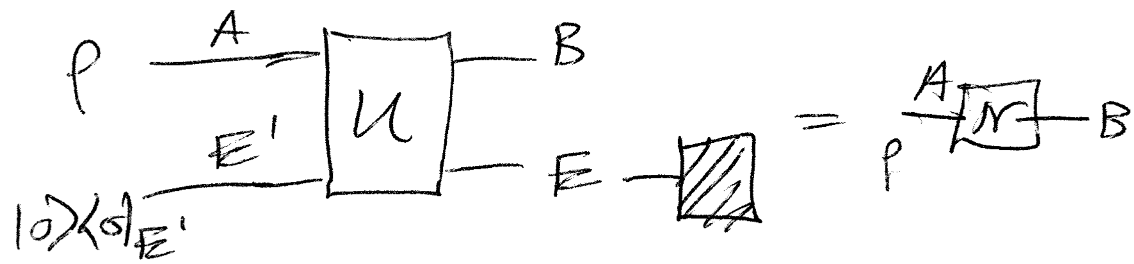
1) tensor in environment state $|0\rangle\langle 0|_{E'}$

2) Apply unitary $U_{AE'\to BE}$ on input system A + environment E'

3) Trace out environment E

$$\mathcal{N}_{A\to B}(X_A) = \mathrm{Tr}_E\left[\mathcal{U}\,(X_A \otimes |0\rangle\langle 0|_{E'})\,\mathcal{U}^\dagger\right]$$

where $\mathcal{U} \equiv \mathcal{U}_{AE'\to BE}$



dimension of $E' \geq \mathrm{rank}(\Gamma^{\mathcal{N}}_{AB})$

$=$ minimal # of Kraus operators needed to realize channel

recall that

$$\mathcal{U}|\psi\rangle|0\rangle_{E'} = V|\psi\rangle$$

where V is an isometry

so we can also write this as

$$\mathcal{N}_{A\to B}(X_A) = \mathrm{Tr}_E\left[V X_A V^\dagger\right]$$

where $V \equiv V_{A\to BE}$

Proof of thm:

Set $V_{A\to BE} = \sum_i K_i \otimes |i\rangle_E$

"canonical construction of isometric extension of the channel"

isometry check

$$V^\dagger V = \left(\sum_i K_i^\dagger \otimes \langle i|_E\right)\left(\sum_j K_j \otimes |j\rangle_E\right)$$

$$= \sum_{ij} K_i^\dagger K_j \otimes \langle i|j\rangle$$

$$= \sum_i K_i^\dagger K_i = I \quad \checkmark$$

extension check

$$\mathrm{Tr}_E[V X V^\dagger] = \mathrm{Tr}_E\left[\left(\sum_i K_i \otimes |i\rangle_E\right) X \left(\sum_j K_j^\dagger \otimes \langle j|_E\right)\right]$$

$$= \mathrm{Tr}_E\left[\sum_{ij} K_i X K_j^\dagger \otimes |i\rangle\langle j|_E\right]$$

$$= \sum_{ij} K_i X K_j^\dagger \otimes \mathrm{Tr}_E[|i\rangle\langle j|_E]$$

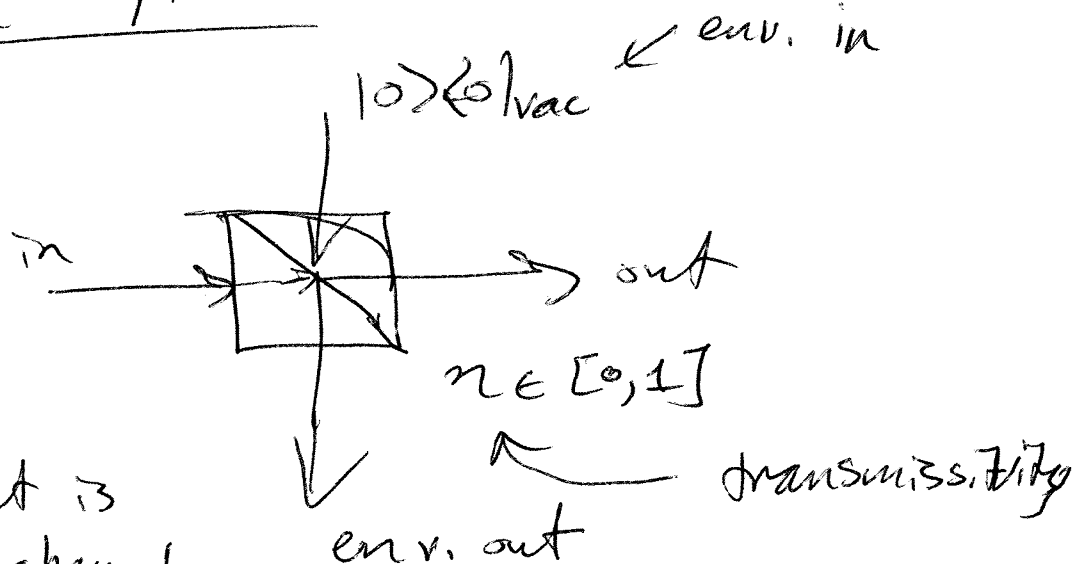
$$= \sum_i K_i X K_i^\dagger = \mathcal{N}(X) \checkmark$$

can realize unitary by setting some entries to

$$\sum_i K_i \otimes |i\rangle_E \langle 0|_{E'}$$

+ filling up the rest to ensure unitarity.

Example of unitary extension from q. optics:



in → out is a loss channel

"Every channel purification (i.e., isometric extension) is related by an isometry acting on purifying system."

I.e., Let $V_{A\to BE}$ be an iso. ext. of $\mathcal{N}_{A\to B}$.

Then $W_{E\to\tilde{E}} V_{A\to BE}$ is also an iso. ext., where W is an isometry

Proof: Suppose $\mathcal{N}(X) = \mathrm{Tr}_E[V X V^\dagger]$

Then $\mathrm{Tr}_{\tilde{E}}[W_{E\to\tilde{E}} V_{A\to BE} X_A (V_{A\to BE})^\dagger (W_{E\to\tilde{E}})^\dagger]$

$= \mathrm{Tr}_E[(W_{E\to\tilde{E}})^\dagger W_{E\to\tilde{E}} V X V^\dagger]$

$= \mathrm{Tr}[V X V^\dagger] = \mathcal{N}(X)$

cyclicity of partial trace

<u>converse part:</u> if $V_{A\to BE}$ +

$V'_{A\to BE'}$ are two different isometric extensions of the same channel $\mathcal{N}_{A\to B}$, then they are related by an isometry $W_{E\to E'}$:

$$V' = WV$$

<u>Examples of channels</u>

Amplitude damping channel

$$\mathcal{A}_\gamma(\rho) = A_1 \rho A_1^\dagger + A_2 \rho A_2^\dagger$$

where $A_1 = \sqrt{\gamma}\, |0\rangle\langle 1|$

$$A_2 = |0\rangle\langle 0| + \sqrt{1-\gamma}\, |1\rangle\langle 1|$$

$\gamma \in [0,1]$ is damping parameter

Models spontaneous emission or loss

$$A_\gamma(|1\rangle\langle 1|) = \gamma |0\rangle\langle 0| + (1-\gamma)|1\rangle\langle 1|$$
$$A_\gamma(|0\rangle\langle 0|) = |0\rangle\langle 0|$$

Erasure channel

$$\mathcal{E}_p(\rho) = (1-p)\rho + p \operatorname{Tr}[\rho] |e\rangle\langle e|$$

where $|e\rangle\langle e|$ is orthogonal to every input ρ

$p \in [0,1]$ is erasure probability.

models heralded loss

both amplitude damping & erasure channels can be induced from pure loss channel

Pauli channels

$$\rho \to p_I \rho + p_X X\rho X + p_Y Y\rho Y + p_Z Z\rho Z$$

where $\{p_I, p_X, p_Y, p_Z\}$ is a prob. dist.

two special instances:

depbasing channel

$$\rho \to (1-p)\rho + p Z\rho Z$$

depolarizing channel

$$\rho \to (1-p)\rho + \frac{p}{3}(X\rho X + Y\rho Y + Z\rho Z)$$

can show that this is the same as

$$\rho \to \left(1-\frac{4p}{3}\right)\rho + \frac{4p}{3}\mathrm{Tr}[\rho]\frac{I}{2}$$

~~More~~ Other Kinds of channels

Preparation channel

$$\mathcal{P}_\rho(\alpha) = \alpha \rho_A$$

where $\alpha \in \mathbb{C}$

prepares the state ρ

Appending channel

$$\mathcal{P}_{\rho_A}(\sigma_B) = (\mathcal{P}_{\rho_A} \otimes \mathrm{id}_B)(\sigma_B)$$

$$= \rho_A \otimes \sigma_B$$

Replacer channel

$$\mathcal{R}^{\sigma_B}_{A\to B}(X_A) = \mathrm{Tr}[X_A]\, \sigma_B$$

$$\mathcal{R}^{\sigma_B}_{A\to B}(X_{RA}) = \mathrm{Tr}_A[X_{RA}] \otimes \sigma_B$$

partial trace & trace are channels also.

Unitary channels

$$\rho \to U \rho U^{\dagger}$$

can be reversed as $\rho \to U^{\dagger} \rho U$

Isometric channels

$$\rho \to V \rho V^{\dagger} \qquad \text{where } V^{\dagger} V = I$$

How to reverse?

$$R_V(Y) = V^{\dagger} Y V + \mathrm{Tr}[(I - V V^{\dagger}) Y]\sigma$$

where σ is a state

Why does this work?

$$(\mathcal{R}_V \circ \mathcal{V})(X)$$

$$= \mathcal{R}_V(VXV^\dagger)$$

$$= V^\dagger(VXV^\dagger)V + \mathrm{Tr}[(I - VV^\dagger)(VXV^\dagger)]\sigma$$

$$= X + \left(\mathrm{Tr}[VXV^\dagger] - \mathrm{Tr}[VV^\dagger VXV^\dagger]\right)\sigma$$

$$= X + \left(\mathrm{Tr}[V^\dagger VX] - \mathrm{Tr}[V^\dagger VV^\dagger VX]\right)\sigma$$

$$= X + \left(\mathrm{Tr}[X] - \mathrm{Tr}[X]\right)\sigma$$

$$= X$$

reverses channel!

can check that

$\mathcal{R}_V$ is CPTP + thus a channel