Lecture 11

Chapter 4 - Quantum Channels

- we already motivated q. channels in an abstract way as a linear, completely positive trace-preserving superoperator acting on states

- Recall the Schrödinger equation for dynamics of state vectors according to a Hamiltonian $H(t)$:

$$i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle = H(t) |\psi(t)\rangle$$

and its generalization by von Neumann:

$$i\hbar \frac{\partial \rho(t)}{\partial t} = [H(t), \rho(t)]$$

for <u>closed</u> quantum systems

solution implies <u>unitary evolution</u>:

$$|\psi(t)\rangle = U(t)\,|\psi_0\rangle$$

&

$$\rho(t) = U(t)\,\rho_0\,U(t)^\dagger$$

if Hamiltonian is time-independent, then $U(t) = e^{-iHt/\hbar}$.

here we are interested in evolution of <u>open</u> quantum systems & thus require the formalism of quantum channels.

Reminder of quantum channels: $\mathcal{N}_{A\to B}$

1) linear

$$\mathcal{N}(\alpha X + \beta Y) = \alpha\,\mathcal{N}(X) + \beta\,\mathcal{N}(Y)$$

$\forall \alpha, \beta \in \mathbb{C}$ & $X, Y \in \mathcal{L}(\mathcal{H}_A)$

2) <u>trace preserving</u>: $\mathrm{Tr}[\mathcal{N}(X)] = \mathrm{Tr}[X]$

$\forall X \in L(H_A)$

3) <u>completely positive</u>:

$(id_R \otimes \mathcal{N}_{A\to B})(X_{RA}) \geq 0$ if

$X_{RA} \geq 0$ for reference system R of arbitrary size.

$$X_{RA} = \sum_{ij} |i\rangle\langle j|_R \otimes X_A^{ij}$$

$$\Rightarrow (id_R \otimes \mathcal{N}_{A\to B})(X_{RA})$$

$$= \sum_{ij} |i\rangle\langle j|_R \otimes \mathcal{N}_{A\to B}(X_A^{ij})$$

Kraus representation of ~~channel~~ superoperator

$$\mathcal{N}(X) = \sum_{i=1}^{n} K_i X L_i^{\dagger}$$

$\mathcal{N}$ is CP iff $L_i = K_i$

$\mathcal{N}$ is TP iff $\sum_i L_i^{\dagger} K_i = I$

Then $\mathcal{N}$ is a channel iff $\mathcal{N}(X) = \sum_i K_i X K_i^{\dagger}$ w/ $\sum_i K_i^{\dagger} K_i = I$

Choi representation of superoperator

$$\Gamma_{AB}^{\mathcal{N}} = \left(id_A \otimes \mathcal{N}_{A' \to B}\right)\left(\Gamma_{AA'}\right)$$

where $\Gamma_{AA'} = \sum_{i,j=0}^{d_A - 1} |i\rangle\langle j|_A \otimes |i\rangle\langle j|_{A'}$

$\mathcal{N}$ is CP iff $\Gamma_{AB}^{\mathcal{N}} \geq 0$

$\mathcal{N}$ is TP iff $\mathrm{Tr}_B\left[\Gamma_{AB}^{\mathcal{N}}\right] = I_A$

Let's investigate some of these conditions:

Suppose $\mathcal{N}_{A\to B}$ is CP.

Then $\Gamma^{\mathcal{N}}_{AB}$ is PSD (by definition of CP)

For the converse, consider the following "post-selected teleportation" identity

$$\mathcal{N}_{A\to B}(X_{RA}) =$$

$$\langle\Gamma|_{AA'} X_{RA} \otimes \Gamma^{\mathcal{N}}_{A'B} |\Gamma\rangle_{AA'}$$

then if $X_{RA} \geq 0$ & $\Gamma^{\mathcal{N}}_{AB} \geq 0$ it follows that $\mathcal{N}_{A\to B}(X_{RA}) \geq 0$ & claim follows.

Why is this identity true?

Consider that

$$\mathcal{N}(|i\rangle\langle j|_A) = \left(\langle i|_A \otimes I_B\right) \Gamma^{\mathcal{N}}_{AB} \left(|j\rangle_A \otimes I_B\right)$$

Expand X_{RA} as

$$\sum_{ij} X_R^{ij} \otimes |i\rangle\langle j|_A$$

By linearity

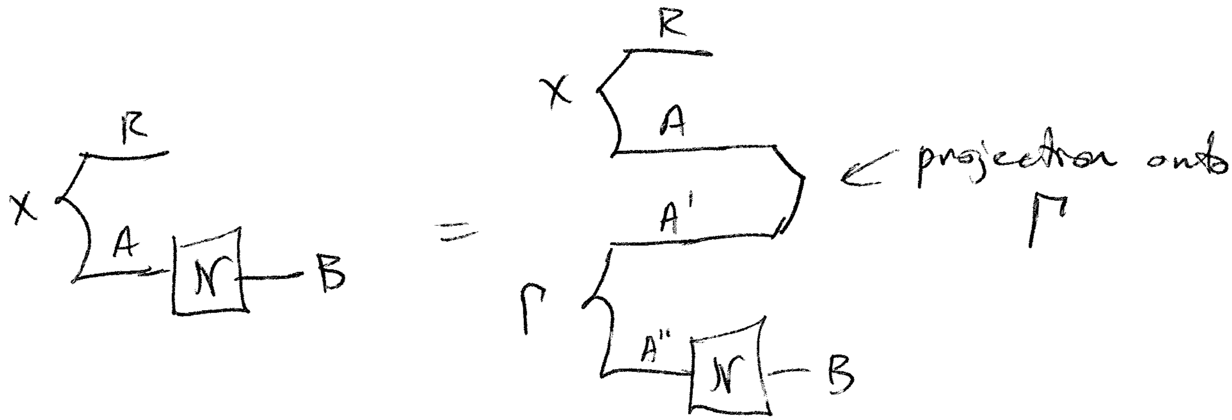
$$\mathcal{N}_{A\to B}(X_{RA}) = \sum_{ij} X_R^{ij} \otimes \mathcal{N}_{A\to B}(|i\rangle\langle j|)$$

$$= \sum_{ij} X_R^{ij} \otimes \langle i|_A \Gamma^{\mathcal{N}}_{AB} |j\rangle_A$$

$$= \sum_k \langle k|_A \langle k|_{A'} \sum_{ij} X_R^{ij} \otimes |i\rangle\langle j|_A \otimes \Gamma^{\mathcal{N}}_{A'B} \sum_l |l\rangle_A |l\rangle_{A'}$$

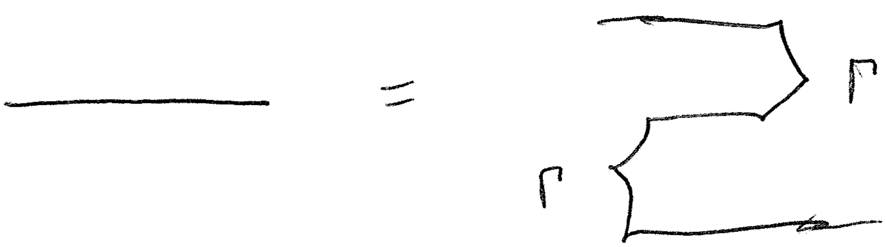
$$= \langle \Gamma|_{AA'} X_{RA} \otimes \Gamma^{\mathcal{N}}_{A'B} |\Gamma\rangle_{AA'}$$

Diagrammatic Notation for this identity:



see it by pulling the line;

more specifically,



this can be used a lot...

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connection of $\Gamma^{\mathcal{N}}_{AB}$ & trace preservation

if $\mathcal{N}$ is TP, then

$$\mathrm{Tr}_B\left[\Gamma^{\mathcal{N}}_{AB}\right] = I_A$$

follows b/c

$$\mathrm{Tr}_B\{\Gamma^{N}_{AB}\} = \mathrm{Tr}_B\Big[\sum_{ij} |i\rangle\langle j|_A \otimes N_{A'\to B}(|i\rangle\langle j|)\Big]$$

$$= \sum_{ij} |i\rangle\langle j|_A \otimes \mathrm{Tr}\big[N(|i\rangle\langle j|)\big]$$

$$\downarrow \text{TP}$$

$$= \sum_{ij} |i\rangle\langle j|_A \otimes \mathrm{Tr}\big[|i\rangle\langle j|\big]$$

$$= \sum_{i} |i\rangle\langle i|_A = I_A .$$

To see the other way, use identity

$$N(X_A) = \langle\Gamma|_{AA'}\, X_A \otimes \Gamma^{N}_{A'B}\, |\Gamma\rangle_{AA'}$$

$$\Rightarrow \mathrm{Tr}[N(X_A)]$$

$$= \mathrm{Tr}_B[\quad \cdots \quad]$$

$$= \langle\Gamma|_{AA'}\, X_A \otimes \mathrm{Tr}_B[\Gamma^{N}_{A'B}]\, |\Gamma\rangle_{AA'}$$

$$= \langle\Gamma|_{AA'}\, X_A \otimes I_{A'}\, |\Gamma\rangle_{AA'}$$

$$= \mathrm{Tr}[X_A] \Rightarrow N \text{ is TP}$$

What about Kraus representation?

Suppose $\mathcal{N}(X) = \sum_i K_i X K_i^\dagger$

then $\mathcal{N}$ is CP b/c

$$\mathcal{N}_{A\to B}(X_{RA}) = \sum_i K^i_{A\to B} X_{RA} (K^i_{A\to B})^\dagger$$

if $X_{RA} \geq 0$, then

$$K^i X K^{i\dagger} \geq 0 \quad \forall i$$

+ sum of these is PSD

+ so $\mathcal{N}$ is CP.

Suppose $\sum_i K_i^\dagger K_i = I$

then $\mathcal{N}$ is TP b/c

$$\mathrm{Tr}[\mathcal{N}(X)] = \mathrm{Tr}\left[\sum_i K_i X K_i^\dagger\right]$$

$$= \sum_i \mathrm{Tr}[K_i X K_i^\dagger]$$

$$= \sum_i \mathrm{Tr}[K_i^\dagger K_i X] = \mathrm{Tr}[\sum_i K_i^\dagger K_i X]$$
$$= \mathrm{Tr}[X]$$

Now suppose that $\mathcal{N}$ is CP.
Let us derive Kraus rep. from this.

Recall that

$$\mathcal{N}_{A\to B}(|i\rangle\langle j|_A) = \langle i|_A \Gamma^{\mathcal{N}}_{AB} |j\rangle_A$$

Consider spectral decomposition of $\Gamma^{\mathcal{N}}_{AB}$ as

$$\Gamma^{\mathcal{N}}_{AB} = \sum_l |\phi^l\rangle\langle\phi^l|_{AB}$$

Recall that $|\phi^l\rangle_{AB} = I_A \otimes K^l_{A'\to B} |\Gamma\rangle_{AA'}$

then $\langle i|_A |\phi^l\rangle_{AB} = \langle i|_A K^l_{A'\to B} |\Gamma\rangle_{AA'}$

$$= K^l_{A'\to B} |i\rangle_{A'}$$

$$\Rightarrow \quad {}_A\langle i| \Gamma^{\mathcal{N}}_{AB} |j\rangle_A$$

$$= \sum_{\ell} {}_A\langle i|\phi^{\ell}\rangle_{AB} \langle \phi^{\ell}|_{AB} |j\rangle_A$$

$$= \sum_{\ell} K^{\ell}_{A' \to B} |i\rangle_{A'} \langle j|_{A'} (K^{\ell}_{A' \to B})^{\dagger}$$

$$\Rightarrow \mathcal{N}(|i\rangle\langle j|) = \sum_{\ell} K^{\ell} |i\rangle\langle j| K^{\ell \dagger}$$

this is Kraus rep.

Suppose $\mathcal{N}$ is TP

Then $\mathrm{Tr}[\mathcal{N}(X)] = \mathrm{Tr}[X] \quad \forall X$

Pick $X = |i\rangle\langle j|$

$$\mathrm{Tr}[\mathcal{N}(|i\rangle\langle j|)] = \mathrm{Tr}[|i\rangle\langle j|]$$
$$= \langle j|i\rangle = \delta_{ij}$$

$$\Rightarrow \quad \mathrm{Tr}[\mathcal{N}(|i\rangle\langle j|)]$$

$$= \mathrm{Tr}\Big[\sum_{\ell} K_\ell |i\rangle\langle j| K_\ell^{+}\Big]$$

$$= \langle j| \sum_{\ell} K_\ell^{+} K_\ell |i\rangle$$

$$= \delta_{ij}$$

$$\Rightarrow \quad \sum_{\ell} K_\ell^{+} K_\ell = I$$