$\boxed{\text{Lecture 9}}$

can generalize Bell states to qudit Bell states, using Heisenberg-Weyl operators $\{W_{z,x} : 0 \le z, x \le d-1\}$

$$W_{z,x} = Z(z)X(x)$$

where

$$Z(z) = \sum_{k=0}^{d-1} e^{2\pi i k z/d} |k\rangle\langle k|$$

$$X(x) = \sum_{k=0}^{d-1} |k+x\rangle\langle k|$$

Qudit Bell states are

$$|\Phi_{z,x}\rangle = (W_{z,x} \otimes I)|\Phi_d\rangle$$

where $|\Phi_d\rangle = \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} |j\rangle|j\rangle$

is canonical maximally entangled state.

Can prove that

$$\langle \Phi_{z',x'} | \Phi_{z,x} \rangle = \delta_{z,z'} \delta_{x,x'}$$

so that $\{|\Phi_{z,x}\rangle\}_{z,x}$ is an orthonormal set.

then
$$\sum_{z,x=0}^{d-1} |\Phi_{z,x}\rangle\langle\Phi_{z,x}| = I \otimes I$$

State purification

Let ρ_A be a state of a system A.

A <u>purification</u> of ρ_A is a pure state $|\psi\rangle\langle\psi|_{RA}$ such that

Thm.: For every state ρ_A,
$\exists$ a purification ψ_{RA} such that $d_R \geq \text{rank}(\rho_A)$

Proof: spectral decomposition:

$$\rho_A = \sum_{k=1}^{r} \lambda_k |\phi_k\rangle\langle\phi_k|_A$$

Now construct

$$|\psi\rangle_{RA} = \sum_{k=1}^{r} \sqrt{\lambda_k}\, |k\rangle_R \otimes |\phi_k\rangle_A$$

can check that

$$\text{Tr}_R\left[|\psi\rangle\langle\psi|_{RA}\right] = \rho_A$$

What happened? changed probabilities $\{\lambda_k\}_k$ to <u>probability amplitudes</u>

can also construct a purification of ρ_A as

$$|\psi\rangle_{RA} = (I_R \otimes \sqrt{\rho_A})\,|\Gamma\rangle_{RA}$$

where $|\Gamma\rangle_{RA} = \sum_{i=0}^{d_A-1} |i\rangle_R |i\rangle_A$

<u>interpretation of purification:</u>

mixedness in a state can be thought of as being due to entanglement w/ an inaccessible reference system,

if $\rho_A = |\phi\rangle\langle\phi|_A$, then only possible purification is

$$|\psi\rangle\langle\psi|_{RA} = |\varphi\rangle\langle\varphi|_R \otimes |\phi\rangle\langle\phi|_A$$

All purifications are related by an isometry acting on reference system:

$$|\psi'\rangle_{R'A} = \left(V_{R\to R'} \otimes I_A\right) |\psi\rangle_{RA}$$

Consider that

$$\mathrm{Tr}_{R'}\left[\left(V_{R\to R'} \otimes I_A\right) |\psi\rangle\langle\psi|_{RA} \left(V_{R\to R'} \otimes I_A\right)^{\dagger}\right]$$

$$= \mathrm{Tr}_R\left[\left((V_{R\to R'})^{\dagger} V_{R\to R'} \otimes I\right) |\psi\rangle\langle\psi|_{RA}\right]$$

$$= \mathrm{Tr}_R\left[|\psi\rangle\langle\psi|_{RA}\right]$$

$$= \rho_A$$

$\Rightarrow$ $|\psi'\rangle$ is purification of ρ_A

can also show via Schmidt decomposition that $|\psi'\rangle_{RA}$ & $|\psi\rangle_{RA}$ are related by isometry

Extension of a state

ω_{RA} is an extension of ρ_A

if $Tr_R[\omega_{RA}] = \rho_A$

(ω_{RA} need not be pure)

Group-Invariant States

- A group G is a tuple $(G, *)$ consisting of a set G an associative operation $*$.
- $*$ is defined so that $g * g' \in G \quad \forall g, g' \in G$.

Also, $\exists$ identity satisfying $id * g = g = g * id \quad \forall g \in G$

- associated w/ every element g is an inverse g^{-1} satisfying

$$g * g^{-1} = g^{-1} * g = id$$

unitary representation of a group is a set $\{U^g\}_{g \in G}$ of unitary operators satisfying $U^g U^{g'} = U^{g * g'}$ $\forall g \in G$.

action of group representation on a quantum system is defined by $\rho \to \cancel{\rho} U^g \rho U^{g\dagger}$ $\forall g \in G$

group invariant state satisfies
$\rho = U^g \rho U^{g\dagger}$ $\forall g \in G$

Examples

"Incoherent" states

cyclic group $\mathbb{Z}_2$ represented by $\{I, Z\}$

group invariant states are

$$p|0\rangle\langle 0| + (1-p)|1\rangle\langle 1| \quad \forall p \in [0,1]$$

permutation symmetric for two qubits

symmetric group S_2 represented by $\{I \otimes I, \text{SWAP}\}$

where $\text{SWAP} = \sum_{ij} |i\rangle\langle j| \otimes |j\rangle\langle i|$

can check that

$|\Phi\rangle\langle\Phi|$ where $|\Phi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$

is group invariant