Lecture 7

Example of an SDP

spectral norm of Hermitian operator H

recall $\|H\|_\infty$ = largest singular value of H

$= \max \{ |\lambda_{max}|, |\lambda_{min}| \}$

Define $f(H) := \sup_{X_1, X_2 \geq 0} \{ \mathrm{Tr}[H(X_1 - X_2)] : \mathrm{Tr}[X_1 + X_2] \leq 1 \}$

$\hat{f}(H) = \inf_{t \geq 0} \{ t : -tI \leq H \leq tI \}$

Theorem: $f(H) = \hat{f}(H) = \|H\|_\infty$

1st show that $f(H)$ can be written as

$$S(\Phi, A, B) = \sup_{X \geq 0} \{ \mathrm{Tr}[AX] : \Phi(X) \leq B \}$$

Pick $X = \begin{bmatrix} X_1 & Z^+ \\ Z & X_2 \end{bmatrix}$,

$$A = \begin{bmatrix} H & 0 \\ 0 & -H \end{bmatrix},$$

$$\Phi(X) = \mathrm{Tr}[X_1 + X_2], \quad B = 1$$

$X \geq 0 \Rightarrow X_1, X_2 \geq 0$

Notice that Z plays no role in the objective nor in the constraint. So it can be eliminated + we can set

$$X = \begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix}$$

③

recall that spectral norm is given by $\|H\|_\infty = \max\{|\lambda_{max}|, |\lambda_{min}|\}$

Let $|\phi_{max}\rangle$ satisfy $H|\phi_{max}\rangle = \lambda_{max}|\phi_{max}\rangle$,

" $|\phi_{min}\rangle$ " $H|\phi_{min}\rangle = \lambda_{min}|\phi_{min}\rangle$.

- Suppose $\lambda_{max} \geq 0$.

  Then choose $X_1 = |\phi_{max}\rangle\langle\phi_{max}|$ & $X_2 = 0$

  $\Rightarrow f(H) \geq \lambda_{max}$

- Suppose $\lambda_{max} \leq 0$.

  Then choose $X_1 = 0$, $X_2 = |\phi_{max}\rangle\langle\phi_{max}|$

  $\Rightarrow f(H) \geq -\lambda_{max} = |\lambda_{max}|$

we conclude that $f(H) \geq |\lambda_{max}|$

Can play a similar game to conclude that

$$f(H) \geq |\lambda_{min}|$$

So we find that $f(H) \geq \max\{|\lambda_{max}|, |\lambda_{min}|\}$

Now let's show that

$$f(H) \leq \hat{f}(H)$$

To do so, we show that $\hat{f}(H)$ is the SDP dual of $f(H)$ & we apply weak duality

5

To find dual SDP, we need to find Φ^+ (this always the main issue when finding the dual.)

Since $\Phi(X) = \mathrm{Tr}[X_1 + X_2]$ & $B = 1$ are scalars, take $Y = t$ to be a scalar.

Then

$$\mathrm{Tr}[Y \Phi(X)] = t\,\mathrm{Tr}[X_1 + X_2]$$

$$= \mathrm{Tr}\left[\begin{pmatrix} tI & 0 \\ 0 & tI \end{pmatrix}\begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix}\right]$$

$$= \mathrm{Tr}[\Phi^+(Y)\, X]$$

$$\Rightarrow \Phi^+(Y) = \Phi^+(t) = \begin{bmatrix} tI & 0 \\ 0 & tI \end{bmatrix}$$

Now plug into standard form of SDP dual:

$$\inf_{Y \geq 0} \left\{ \operatorname{Tr}[BY] : \Phi^{\dagger}(Y) \geq A \right\}$$

$$= \inf_{t \geq 0} \left\{ t : \begin{bmatrix} tI & 0 \\ 0 & tI \end{bmatrix} \geq \begin{bmatrix} H & 0 \\ 0 & -H \end{bmatrix} \right\}$$

$$= \inf_{t \geq 0} \left\{ t ; tI \geq H, \; tI \geq -H \right\}$$

$$= \inf_{t \geq 0} \left\{ t : -tI \leq H \leq tI \right\}$$

$$= \hat{f}(H)$$

Consider that

$$\lambda_{min} I \leq H \leq \lambda_{max} I$$

& also that $\lambda_{max} I \leq \|H\|_\infty I$

& $-\|H\|_\infty I \leq \lambda_{min} I$

$$-\|H\|_\infty I \preceq H \preceq \|H\|_\infty I$$

$\Rightarrow$ $\|H\|_\infty$ is a feasible choice for t in $\hat{f}(H)$

$$\Rightarrow \quad \hat{f}(H) \leq \|H\|_\infty$$

Putting everything together, we get

$$\|H\|_\infty \underset{\substack{\uparrow \\ \text{1st part} \\ \text{of proof}}}{\leq} f(H) \underset{\substack{\uparrow \\ \text{weak} \\ \text{duality}}}{\leq} \hat{f}(H) \underset{\substack{\uparrow \\ \text{2nd} \\ \text{part of proof}}}{\leq} \|H\|_\infty$$

done...

Can evaluate complementary slackness conditions & they reduce to

(Recall
$YB = Y\Phi(X)$
$\Phi^\dagger(Y)X = AX$)

$$t = t\, \mathrm{Tr}[X_1 + X_2]$$

$$\begin{bmatrix} tI & 0 \\ 0 & tI \end{bmatrix}\begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix} = \begin{bmatrix} H & 0 \\ 0 & -H \end{bmatrix}\begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix}$$

$$\Leftrightarrow \quad tX_1 = HX_1 \quad -tX_2 = HX_2$$

If we know eigenvalues of H are non-zero, then we conclude that

$$t \neq 0 \quad \& \quad \mathrm{Tr}[X_1 + X_2] = 1$$

So it suffices to optimize over X_1 & X_2 satisfying

other conditions imply that

X_1 & X_2 commute w/ H.

Bring up Matlab code...