# Lecture 6

Semidefinite programming (SDP)
AKA semidefinite optimization

important class of optimization problems that arise in QIT.

Many bounds on rates of communication can be phrased as SDPs

- useful as numerical & analytical tool (perhaps one of the most important tools in modern QIT)

$\Phi$ is a Hermiticity preserving superoperator if $\Phi(X)$ is Hermitian whenever $X$ is.

SDP: Input: Hermitian operators $A$ & $B$ & Hermiticity preserving superop. $\Phi$

primal SDP: $\sup_{X \geq 0} \{ \mathrm{Tr}[AX] : \Phi(X) \leq B \} \equiv S(\Phi, A, B)$

$X \geq 0$ & $\Phi(X) \leq B$ are operator inequalities, meaning that $X$ is PSD & $B - \Phi(X)$ is PSD

directly relevant to QIT, we'll learn that states satisfy $X \geq 0$ & $\mathrm{Tr}[X] = 1$ & measurement operators satisfy $\Lambda \geq 0$ $\Lambda \leq I$

related to the primal SDP, there is a <u>dual SDP</u>, in which the objective function & constraint are "flipped" as well as the optimization

dual SDP:

$$\hat{S}(\Phi, A, B) \equiv \inf_{Y \geq 0} \left\{ \mathrm{Tr}[BY] : \Phi^{+}(Y) \geq A \right\}$$

$\Phi^{+}$ is adjoint of $\Phi$

(i.e., $\mathrm{Tr}[Y^{+} \Phi(X)] = \mathrm{Tr}[[\Phi^{+}(Y)]^{+} X] \quad \forall X, Y$)

---

$X$ is called <u>primal feasible</u> if $\Phi(X) \leq B$ & $X \geq 0$

$Y$ is <u>dual feasible</u> if $\Phi^{+}(Y) \geq A$ & $Y \geq 0$

There may not exist primal feasible or dual feasible operators.

If no primal feasible exists, then

$$S(\Phi, A, B) = -\infty$$

If no dual feasible, then

$$\hat{S}(\Phi, A, B) = +\infty$$

---

<u>Weak duality</u>

$$S(\Phi, A, B) \leq \hat{S}(\Phi, A, B)$$

always holds!

proof is simple:

Let $X, Y \geq 0$ be feasible.

then $\mathrm{Tr}[AX] \leq \mathrm{Tr}[\Phi^*(Y) X] = \mathrm{Tr}[Y \Phi(X)]$
$\leq \mathrm{Tr}[YB]$

Since these inequalities hold $\forall$ feasible $X, Y$

$\Rightarrow$

$$\sup_{X \text{ primal feasible}} \mathrm{Tr}[AX] \leq \inf_{Y \text{ dual feas.}} \mathrm{Tr}[BY]$$

$\blacksquare$

---

connection between weak duality & max-min inequality

$$\left( \sup_Y \inf_X F(X,Y) \leq \inf_X \sup_Y F(X,Y) \right.$$

Consider Lagrangian

$$\mathcal{L} \equiv \mathcal{L}(\Phi, A, B, X, Y)$$
$$= \mathrm{Tr}[AX] + \mathrm{Tr}[BY] - \mathrm{Tr}[\Phi(X)Y]$$

can rewrite as

$$\mathcal{L} = \mathrm{Tr}[AX] + \mathrm{Tr}[(B - \Phi(X))Y]$$

$$= \mathrm{Tr}[BY] + \mathrm{Tr}[(A - \Phi^{\dagger}(Y))X]$$

First take inf over $Y \geq 0$ & then sup over $X \geq 0$

$$\sup_{X \geq 0} \inf_{Y \geq 0} \mathcal{L}$$

$$\Rightarrow = \sup_{X \geq 0} \inf_{Y \geq 0} \mathrm{Tr}[AX] + \mathrm{Tr}[(B - \Phi(X))Y]$$

$$(*) = \sup_{X \geq 0} \left( \mathrm{Tr}[AX] + \inf_{Y \geq 0} \mathrm{Tr}[(B - \Phi(X))Y] \right)$$

focus on this term

Suppose that $B \not\geq \Phi(X)$ (constraint violated) then $B - \Phi(X)$ not PSD

& $\exists\ |\psi\rangle$ such that

$$\langle\psi|\, B - \Phi(X)\, |\psi\rangle < 0$$

Pick $Y = c\, |\psi\rangle\langle\psi|$

for $c > 0$

& then $\mathrm{Tr}[(B - \Phi(X))\, Y]$

$= \mathrm{Tr}[(B - \Phi(X))\, c\, |\psi\rangle\langle\psi|]$

$= c\, \langle\psi|\, B - \Phi(X)\, |\psi\rangle$

By taking $c \to +\infty$, we get

$$\inf_{Y \geq 0} \mathrm{Tr}[(B - \Phi(X))Y] = -\infty$$

So we find that

$$(*) = \sup_{X \geq 0} \{ \mathrm{Tr}[AX] : \Phi(X) \leq B \}$$

$$= S(\Phi, A, B)$$

Similarly,

$$\inf_{Y \geq 0} \sup_{X \geq 0} \mathcal{L} = \hat{S}(\Phi, A, B)$$

Since $\sup_{X \geq 0} \inf_{Y \geq 0} \mathcal{L} \leq \inf_{Y \geq 0} \sup_{X \geq 0} \mathcal{L}$

this is the same as weak duality

<u>strong duality</u> is when the opposite inequality holds

can check using <u>Slater's condition</u>

conclude that

$S(\Phi, A, B) = \hat{S}(\Phi, A, B)$ in this case

## Slater's condition

sufficient condition for strong duality

If $\exists X \geq 0$ such that $\Phi(X) \leq B$

& $\exists Y > 0$ such that

$$\Phi^{\dagger}(Y) > A$$

then $S = \hat{S}$ & $\exists$ primal feasible

$X$ for which $\mathrm{Tr}\{AX\} = S(\Phi, A, B)$

alternative condition to check

w/ non-strict & strict inequalities switched.

---

## Complementary Slackness

helpful for understanding constraints on optimal $X$ & $Y$.

~~For every~~

Consider an SDP corresp. to $\Phi$, $A$, $B$ + suppose strong duality holds.

Then the following conditions hold for feasible $X$ + $Y$ iff they are optimal:

$$YB = Y\Phi(X)$$

$$\Phi^{+}(Y)\cancel{X} = AX$$

Proof: Recall inequalities for weak duality

$$\mathrm{Tr}[AX] \leq \mathrm{Tr}[\Phi^{+}(Y)X] = \mathrm{Tr}[Y\Phi(X)] \leq \mathrm{Tr}[BY]$$

if strong duality holds for optimal $X$ + $Y$, then these are all equalities.

11

$$\Rightarrow \mathrm{Tr}\left[\left(\Phi^{+}(Y) - A\right) X\right] = 0$$

$$\text{\& } \mathrm{Tr}\left[\left(B - \Phi(X)\right) Y\right] = 0$$

since operators involved are PSD

then conclude

$$\left(\Phi^{+}(Y) - A\right) X = 0 \quad \text{\&}$$

$$\left(B - \Phi(X)\right) Y = 0 \quad \Rightarrow \text{statement}$$

Now suppose c.s. equations hold.

Then inequalities are saturated

& operators are thus optimal.