Lecture 4

Mathematical Background & Tools

Finite-dimensional Hilbert spaces (AKA complex Euclidean spaces)

Use $|\psi\rangle$ to denote a vector in $\mathcal{H}$

Hilbert space

Inner product $\langle \cdot | \cdot \rangle : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$:

1) non-negativity: $\langle \psi | \psi \rangle \geq 0 \quad \forall |\psi\rangle \in \mathcal{H}$

& $\langle \psi | \psi \rangle = 0$ iff $|\psi\rangle = 0$

2) conjugate bilinearity

$$\langle \alpha_1 \psi_1 + \beta_1 \phi_1 | \alpha_2 \psi_2 + \beta_2 \phi_2 \rangle$$

$$= \bar{\alpha}_1 \alpha_2 \langle \psi_1 | \psi_2 \rangle + \bar{\alpha}_1 \beta_2 \langle \psi_1 | \phi_2 \rangle + \bar{\beta}_1 \alpha_2 \langle \phi_1 | \psi_2 \rangle + \bar{\beta}_1 \beta_2 \langle \phi_1 | \phi_2 \rangle$$

$\forall |\psi_1\rangle, |\psi_2\rangle, |\phi_1\rangle, |\phi_2\rangle \in \mathcal{H}$ & $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{C}$

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3) conjugate symmetry

$$\langle\psi|\phi\rangle = \overline{\langle\phi|\psi\rangle} \quad \forall\ |\psi\rangle, |\phi\rangle \in \mathcal{H}$$

$\bar{\alpha}$ denotes complex conjugate of $\alpha \in \mathbb{C}$.

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take $|0\rangle = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$, $|1\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$, ...

$$|d-1\rangle = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

$$\langle i|j\rangle = \delta_{ij}$$

Define inner product to be

$$\langle\psi|\phi\rangle = \sum_{i=0}^{d-1} \bar{\alpha}_i \beta_i \quad \text{for}$$

$|\psi\rangle = \sum_i \alpha_i |i\rangle$ & $|\phi\rangle = \sum_i \beta_i |i\rangle$

Euclidean norm:

$$\| |\psi\rangle \|_2 = \sqrt{\langle\psi|\psi\rangle}$$

Cauchy - Schwarz:

$$|\langle\psi|\phi\rangle|^2 \leq \langle\psi|\psi\rangle \cdot \langle\phi|\phi\rangle$$

w/ equality iff $\exists\, \alpha \in \mathbb{C}$ s.t. $|\phi\rangle = \alpha|\psi\rangle$.

$\langle\psi|$ is dual vector of $|\psi\rangle$,

linear functional from $\mathcal{H}$ to $\mathbb{C}$ such that

$$\langle\psi|\left(|\phi\rangle\right) = \langle\psi|\phi\rangle \quad \forall\, |\phi\rangle \in \mathcal{H}.$$

can express as $\langle\psi| = \sum_i \bar{\alpha}_i \langle i|$

where $\langle i| = (0 \cdots 0\,1\,0 \cdots 0)$ (row vector)

$(i+1)^{th}$ entry

$|i\rangle_A \otimes |j\rangle_B$ is tensor product

of $|i\rangle_A \in \mathcal{H}_A$ + $|j\rangle_B \in \mathcal{H}_B$

Example:

$i=0$ + $j=2$

$$|0\rangle \otimes |2\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \\ 0 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

More generally,

$$|\psi\rangle_A \otimes |\phi\rangle_B = \sum_i^1 \sum_j^2 \alpha_i \beta_j \, |i\rangle_A \otimes |j\rangle_B$$

for $|\psi\rangle = \sum_i^1 \alpha_i |i\rangle$

More general Example:

$|\phi\rangle = \sum_j^2 \beta_j |j\rangle$

$$|\psi\rangle \otimes |\phi\rangle = \begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix} \otimes \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} \alpha_0 \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{pmatrix} \\ \alpha_1 \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{pmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} \alpha_0 \beta_0 \\ \alpha_0 \beta_1 \\ \alpha_0 \beta_2 \\ \alpha_1 \beta_0 \\ \alpha_1 \beta_1 \\ \alpha_1 \beta_2 \end{pmatrix}$$

tensor-product Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$ defined as

$$\text{span}\left\{ |i\rangle_A \otimes |j\rangle_B : \begin{array}{l} i \in \{0, \ldots, d_A - 1\}, \\ j \in \{0, \ldots, d_B - 1\} \end{array} \right\}$$

direct sum of $\mathcal{H}_A$ & $\mathcal{H}_B$:

$$\mathcal{H}_A \oplus \mathcal{H}_B$$

defined as Hilbert space of vectors

$$|\psi\rangle_A \oplus |\phi\rangle_B = \begin{pmatrix} |\psi\rangle_A \\ |\phi\rangle_B \end{pmatrix}$$

Linear Operators

$X : \mathcal{H}_A \to \mathcal{H}_B$ is a function satisfying

$$X(\alpha|\psi\rangle_A + \beta|\phi\rangle_A)$$
$$= \alpha X|\psi\rangle_A + \beta X|\phi\rangle_A$$

$\forall \alpha, \beta \in \mathbb{C}$ & $|\psi\rangle_A, |\phi\rangle_A \in \mathcal{H}_A$

can write as $X_{A \to B}$

can identify with matrices
through a matrix representation

Denote set of all linear operators by

$$L(\mathcal{H}_A, \mathcal{H}_B)$$

if input & output spaces the same, then write

$$L(\mathcal{H}_A) \text{ for short.}$$

Consider that

$$X_{A\to B} = I_B \, X_{A\to B} \, I_A$$
$$= \sum_i |i\rangle\langle i|_B \, X_{A\to B} \sum_j |j\rangle\langle j|_A$$
$$= \sum_{ij} |i\rangle_B \langle i|_B X_{A\to B} |j\rangle_A \langle j|_A$$
$$= \sum_{ij} x_{ij} \, |i\rangle_B \langle j|_A$$

where $x_{ij} = \langle i|X|j\rangle$

the set $\{X_{ij}\}_{ij}$ is the matrix representation of $X_{A\to B}$ in the standard basis.

tensor products of linear operators

$$(X \otimes Y)(|\psi\rangle \otimes |\phi\rangle)$$

$$= X|\psi\rangle \otimes Y|\phi\rangle$$

matrix rep. of $X \otimes Y$ is Kronecker product of individual matrix rep's of X & Y:

$$X \otimes Y = \begin{pmatrix} X_{0,0} & X_{0,1} \\ X_{1,0} & X_{1,1} \end{pmatrix} \otimes \begin{pmatrix} Y_{0,0} & Y_{0,1} & Y_{0,2} \\ Y_{1,0} & Y_{1,1} & Y_{1,2} \\ Y_{2,0} & Y_{2,1} & Y_{2,2} \end{pmatrix}$$

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$$= \begin{pmatrix} X_{0,0} \cdot \begin{pmatrix} Y_{0,0} & \\ & \ddots \end{pmatrix} & X_{0,1} \cdot \begin{pmatrix} Y_{0,0} & \\ & \ddots \end{pmatrix} \\ X_{1,0} \cdot \begin{pmatrix} Y_{0,0} & \cdots \\ \vdots & \ddots \end{pmatrix} & X_{1,1} \begin{pmatrix} Y_{0,0} & \cdots \\ \vdots & \ddots \end{pmatrix} \end{pmatrix}$$

Image of operator $X_{A \to B}$:

$$\text{im}(X) = \{ |\phi\rangle_B \in \mathcal{H}_B : |\phi\rangle_B = X |\psi\rangle_A, \; |\psi\rangle_A \in \mathcal{H}_A \}$$

$$\text{rank}(X) = \dim(\text{im}(X))$$

kernel of operator $X_{A \to B}$:

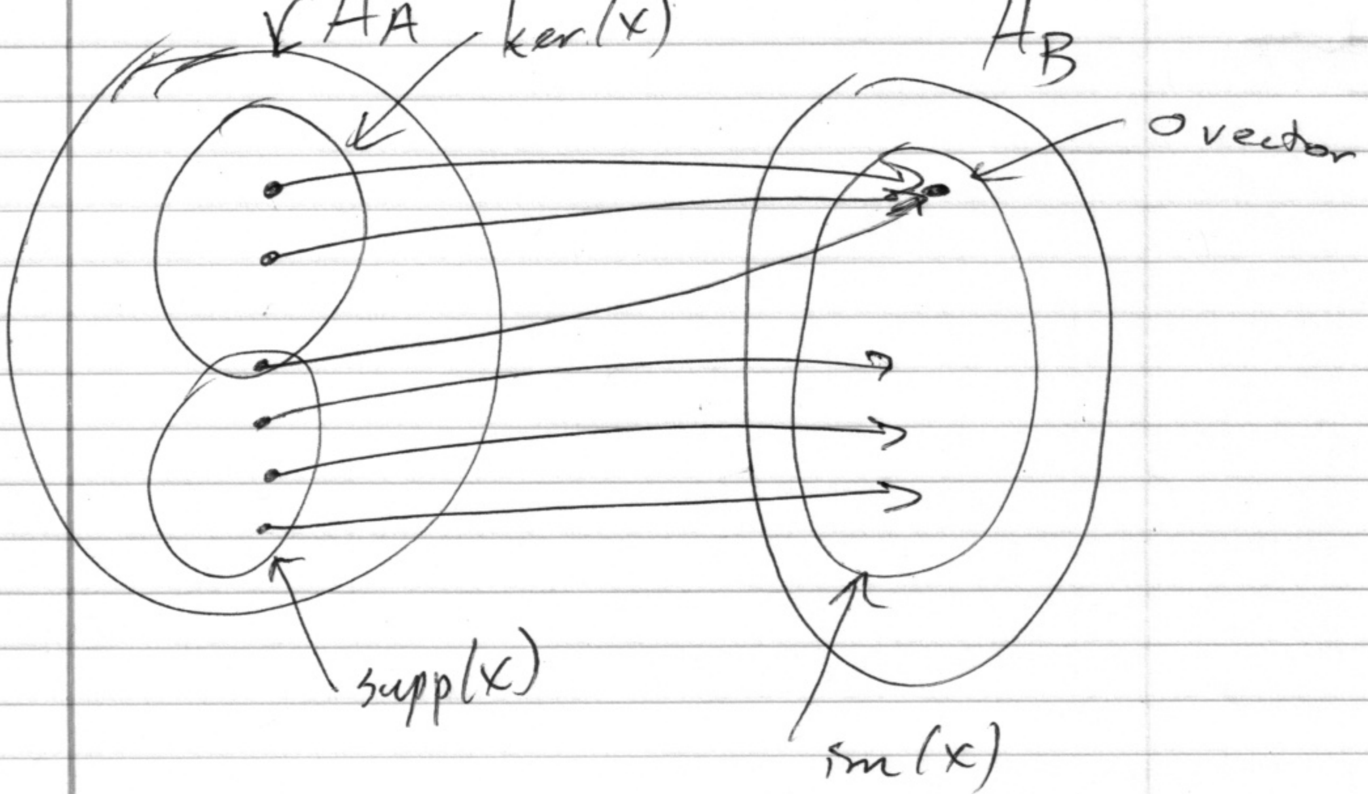
$$\ker(X) = \{ |\psi\rangle_A \in \mathcal{H}_A : X|\psi\rangle_A = 0 \}$$

$$d_A = \text{rank}(X) + \dim(\ker(X))$$

support of X: (orthogonal complement of kernel)

$$\mathrm{supp}(X) = \{ |\psi\rangle \in \mathcal{H}_A : \langle\psi|\phi\rangle = 0 \ \ \forall |\phi\rangle \in \ker(X) \}$$

<u>Picture:</u>



trace:

$$\mathrm{Tr}[X] = \sum_i \langle i|X|i\rangle$$

can use any orthonormal basis

Cyclicity of trace:

$$\mathrm{Tr}[XYZ] = \mathrm{Tr}[ZXY]$$
$$= \mathrm{Tr}[YZX]$$

transpose (basis dependent operation)

X^T or $T(X)$

$$= \sum_{i,j} X_{ij} |j\rangle\langle i|$$

flips $|i\rangle\langle j|$ to $|j\rangle\langle i|$

can be understood as superoperator

$$T(X) = \sum_{ij} \underbrace{|j\rangle\langle i|}\ X\ \underbrace{|j\rangle\langle i|}$$

these are Kraus operators of superoperator.

conjugate transpose (basis independent)

$$X^{+} = \sum_{ij} \overline{X_{ij}}\ |j\rangle\langle i|$$

take transpose & complex conjugate

Hilbert - Schmidt inner product;

$$\langle X, Y \rangle = \mathrm{Tr}[X^\dagger Y]$$

Hilbert - Schmidt norm;

$$\|X\|_2 = \sqrt{\langle X, X \rangle}$$

For every vector $|\psi\rangle_{AB} \in \mathcal{H}_A \otimes \mathcal{H}_B$

$\exists$ $X_{A \to B}$ such that

$$|\psi\rangle_{AB} = (I_A \otimes X_{A \to B})\, |\Gamma\rangle_{AA}$$

where

$$|\Gamma\rangle_{AA} = \sum_i |i\rangle_A \otimes |i\rangle_A$$

called "maximally entangled vector"

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Suppose $|\psi\rangle_{AB} = \sum_{ij} \alpha_{ij} |i\rangle_A \otimes |j\rangle_B$

Then set $X_{A\to B} = \sum_{ij} \alpha_{ij} |j\rangle_B \langle i|_A$

Why does this work?

$$\left(I_A \otimes X_{A\to B}\right)|\Gamma\rangle_{AA}$$

$$= \left(I_A \otimes \sum_{ij} \alpha_{ij} |j\rangle_B \langle i|_A\right)\left(\sum_k |k\rangle_A \otimes |k\rangle_A\right)$$

$$= \sum_{ijk} \alpha_{ij} \; |k\rangle_A \otimes |j\rangle_B \langle i|k\rangle_A$$

$$= \sum_{ij} \alpha_{ij} \; |i\rangle_A \otimes |j\rangle_B$$

$$= |\psi\rangle_{AB}$$

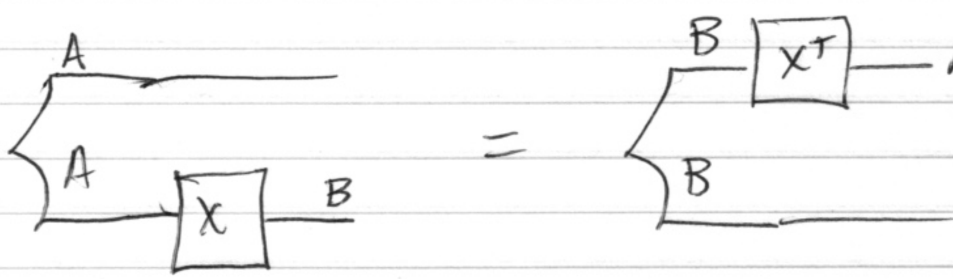
Transpose Trick:

For every $X_{A\to B} \in L(H_A, H_B)$,

$$(I_A \otimes X_{A\to B})\,|\Gamma\rangle_{AA} = (X^T)_{B\to A} \otimes I_B\,|\Gamma\rangle_{BB}$$

notice different dimensions

Interpretation:



Bottom party performing X is the same as top party doing X^T

Different kinds of operators:

1) Hermitian: $X = X^\dagger$

2) Positive semi-definite:

Hermitian & satisfies

$$\langle\psi|X|\psi\rangle \geq 0 \quad \forall |\psi\rangle$$

We write $X \geq Z$ if

$X - Z$ is positive semidefinite

3) positive definite:

$$\langle\psi|X|\psi\rangle > 0 \quad \forall |\psi\rangle$$

↑ strict ineq.

4) density operators:

positive semi-definite

w/ trace one

5) unitary operators :

$U \in L(\mathcal{H})$ & $UU^{+} = U^{+}U = I$

6) Isometries :

$V \in L(\mathcal{H}_A, \mathcal{H}_B)$ &

$$V^{+}V = I_A$$

7) Projections ;

$P \in L(\mathcal{H})$, Hermitian &

satisfies $P^2 = P$

Singular value decomposition:

For every $X \in L(\mathcal{H}_A, \mathcal{H}_B)$

w/ $r = \mathrm{rank}(X)$

$\exists \ \{s_k\}_{k=1}^{r}$ s.t. $s_k > 0 \ \forall k$

↑ singular values

& $\{|e_k\rangle_B\}_{k=1}^{r}$ & $\{|f_k\rangle_A\}_{k=1}^{r}$

orthonormal sets

s.t.

$$X = \sum_{k=1}^{r} s_k |e_k\rangle_B \langle f_k|_A$$

Schmidt Decomposition

For every $|\psi\rangle_{AB} \in \mathcal{H}_A \otimes \mathcal{H}_B$,

we have

$$|\psi\rangle_{AB} = \sum_{k=1}^{r} \sqrt{\lambda_k}\, |e_k\rangle_A \otimes |f_k\rangle_B$$

where $\{\lambda_k\}_{k=1}^{r}$ & $\lambda_k > 0$

are Schmidt coefficients

& $\{|e_k\rangle_A\}_{k=1}^{r}$ & $\{|f_k\rangle_B\}_{k=1}^{r}$

are orthonormal sets

Proof: write $|\psi\rangle_{AB}$ as

$$|\psi\rangle_{AB} = \sum_{ij} \alpha_{ij}\, |i\rangle_A |j\rangle_B$$

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Define operator

$$X = \sum_{ij} \alpha_{ij} |j\rangle_B \langle i|_A \quad (\text{flip } |i\rangle_A \text{ to } \langle i|_A)$$

apply SVD to X to get

$$X = \sum_{k=1}^{r} \sqrt{\lambda_k} \, |f_k\rangle_B \langle g_k|_A$$

then flip back to get

$$|\psi\rangle_{AB} = \sum_{k=1}^{r} \sqrt{\lambda_k} \, \overline{|g_k\rangle_A} \otimes |f_k\rangle_B$$

define $|e_k\rangle_A$ to be $\overline{|g_k\rangle_A}$

Polar decomposition:

every $X \in L(\mathcal{H})$ can be written

as $X = UP$

where U is unitary & P is positive semi-definite

Proof: apply SVD.

Spectral theorem:

Def'n: operator X is normal if $XX^\dagger = X^\dagger X$

For every normal $X \in L(\mathcal{H})$, $\exists\, n \in \mathbb{N}$ s.t.

$$X = \sum_{j=1}^{n} \lambda_j \Pi_j$$

where $\{\lambda_j\}_{j=1}^{n}$ are distinct eigenvalues

&

$\{\Pi_j\}_{j=1}^n$ is set of spectral projections

satisfying $\sum_{j=1}^n \Pi_j = I$

& $\Pi_i \Pi_j = \delta_{ij} \Pi_i$

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can write Hermitian X in terms of positive part & negative part

$$X = X_+ - X_-$$

where $X_+ X_- = 0$

&

$$X_+ = \sum_{k : \lambda_k \geq 0} \lambda_k \Pi_k$$

$$X_- = \sum_{k : \lambda_k < 0} |\lambda_k| \Pi_k$$