1 Overview

In the last lecture, we represented faithful Gaussian states as thermal states of quadratic Hamiltonians and discussed the Williamson theorem.

In this lecture, we first review a method to find the symplectic eigenvalues of a positive definite matrix. We then derive a relation between the Hamiltonian matrix and the covariance matrix corresponding to a faithful Gaussian state. Finally, we determine formulas for the purity and von Neumann entropy of a Gaussian state. We point readers to [Ser17] for background on some of the topics covered in this lecture.

2 Symplectic eigenvalues of a positive definite matrix

In this section, we discuss a method to find the symplectic eigenvalues of a positive definite matrix $M$.

Let $\Omega$ denote the real, canonical, anti-symmetric form defined as

$$\Omega = I_n \otimes \Omega_1,$$

where

$$\Omega_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

which encodes the canonical commutation relations of the quadrature operators. Note that $\Omega\Omega^T = -\Omega^2 = I$.

Let $S$ denote a symplectic matrix such that $S\Omega S^T = \Omega$. It follows that such a matrix $S$ is invertible with inverse given by $S^{-1} = \Omega S^T \Omega^T$. We begin by showing that $\Omega S = S^{-T} \Omega$. This is a direct consequence of the fact that $S^T$ is symplectic, which can be seen from the following steps:

$$S\Omega S^T = \Omega,$$

$$\Rightarrow S\Omega S^T \Omega = -I,$$

$$\Rightarrow S\Omega S^T \Omega S = -S,$$

$$\Rightarrow S^{-1} \Omega S^T \Omega S = -S^{-1} S,$$

$$\Rightarrow \Omega S^T \Omega S = -I,$$

$$\Rightarrow S^T \Omega S = \Omega.$$
It then follows that

$$\Omega S = S^{-T}\Omega.$$  \hspace{1cm} (9)

As discussed in the previous lecture, a positive definite $2n \times 2n$ matrix $M$ has the following symplectic decomposition:

$$M = SDS^T,$$ \hspace{1cm} (10)

where $d_j > 0$ for all $j \in \{1, \ldots, n\}$,

$$D = \bigoplus_{j=1}^{n} d_j \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = D_n \otimes I_2,$$ \hspace{1cm} (11)

and

$$D_n = \text{diag}(d_1, d_2, \ldots, d_n).$$ \hspace{1cm} (12)

We now establish a connection between the symplectic eigenvalues of a positive definite matrix $M$ and the eigenvalues of the matrix $i\Omega M$. Consider the following chain of equalities:

$$i\Omega M = i\Omega SDS^T$$

$$= i\Omega S(D_n \otimes I_2)S^T$$

$$= S^{-T}(i\Omega)(D_n \otimes I_2)S^T$$

$$= S^{-T}(I_n \otimes i\Omega_1)(D_n \otimes I_2)S^T$$

$$= S^{-T}(D_n \otimes i\Omega_1)S^T$$

$$= S^{-T}(D_n \otimes -\sigma_Y)S^T$$

$$= S^{-T}(I_n \otimes U_2)(D_n \otimes -\sigma_Z)(I_n \otimes U_2^\dagger)S^T.$$ \hspace{1cm} (13)-\hspace{1cm} (19)

The first equality follows from (10). The second equality follows from (11). The third equality follows from (9). The fourth equality follows from the definition of $\Omega$ as defined in (1). The last two equalities follow from the fact that

$$i\Omega_1 = -\sigma_Y = U_2(-\sigma_Z)U_2^\dagger,$$ \hspace{1cm} (20)

where

$$U_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}.$$ \hspace{1cm} (21)

From (19) and from the fact that $D_n \otimes -\sigma_Z = \text{diag}(-d_1, d_1, -d_2, d_2, \ldots, -d_n, d_n)$, it follows that the usual eigendecomposition of $i\Omega M$ is given by $B(D_n \otimes -\sigma_Z)B^{-1}$, where

$$B = S^{-T}(I_n \otimes U_2)$$ \hspace{1cm} (22)

is the matrix of eigenvectors. We note that $S^{-T}$ can be expressed in terms of $\Omega$ and $S$. Since $S\Omega S^T = \Omega$, it follows that $S\Omega S^T\Omega^T = \Omega \Omega^T$. Since $\Omega \Omega^T = I$, $S^{-1} = \Omega S^T \Omega^T$. Therefore, $S^{-T} = \Omega S \Omega^T$. 

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Therefore, a method to find the symplectic eigenvalues of a positive definite matrix \( M \) is as follows. We first find the usual eigendecomposition of the matrix \( i\Omega M \), and the corresponding eigenvalues provide the information of symplectic eigenvalues of the matrix \( M \). Moreover, the symplectic matrix \( S \) corresponding to the transformation \( M = SDS^T \), can be found from the eigenvector matrix \( B = S^{-T}(I_n \otimes U_2) = \Omega S \Omega^T (I_n \otimes U_2) \) as defined in (22), i.e., \( S = B^{-T} (I_n \otimes U_2^T) = \Omega^T B (I_n \otimes U_2^T) \Omega \).

3 Relationship between the Hamiltonian matrix and the covariance matrix for a faithful Gaussian state

In this section, we derive the following relations between the Hamiltonian matrix and the covariance matrix corresponding to a faithful Gaussian state:

\[
\sigma = \coth \left( \frac{i\Omega H}{2} \right) i\Omega, \tag{23}
\]
\[
H = 2 \arccoth (i\Omega \sigma) i\Omega. \tag{24}
\]

As discussed in the previous lecture, a positive definite matrix \( H \) can be represented in the following symplectic diagonalized form:

\[
H = S^T \bigoplus_{j=1}^{n} \lambda_j \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} S, \tag{25}
\]

where \( \lambda_j > 0, \forall j \in \{1, \ldots, n\} \).

Moreover, the corresponding covariance matrix \( \sigma \) can be written as

\[
\sigma = S^{-1} \bigoplus_{j=1}^{n} \coth \left( \frac{\lambda_j}{2} \right) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} S^{-T}, \tag{26}
\]

where \( \nu_j \equiv \coth(\lambda_j/2) \) for \( j \in \{1, \ldots, n\} \) are the symplectic eigenvalues of \( \sigma \).

From (19) and (25), it follows that

\[
\frac{1}{2} i\Omega H = \frac{1}{2} S^{-1}(I_n \otimes U_2)(D_n \otimes -\sigma_Z)(I_n \otimes U_2^\dagger)S, \tag{27}
\]

where \( D_n = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n) \).

Consider the following chain of equalities:

\[
\coth \left( \frac{i\Omega H}{2} \right) = S^{-1}(I_n \otimes U_2) \coth \left( \frac{D_n \otimes -\sigma_Z}{2} \right) (I_n \otimes U_2^\dagger)S \tag{28}
\]
\[
= S^{-1}(I_n \otimes U_2)(\coth(D_n/2) \otimes -\sigma_Z)(I_n \otimes U_2^\dagger)S \tag{29}
\]
\[
= S^{-1}(\coth(D_n/2) \otimes i\Omega_1)S \tag{30}
\]
\[
= S^{-1}(\coth(D_n/2) \otimes I_Z)(I_n \otimes i\Omega_1)S \tag{31}
\]
\[
= S^{-1}(\coth(D_n/2) \otimes I_Z)i\Omega S \tag{32}
\]
\[
= S^{-1}(\coth(D_n/2) \otimes I_Z)S^{-T}i\Omega \tag{33}
\]
\[
= \sigma i\Omega. \tag{34}
\]
The first equality follows from (27). The second equality follows from the fact that \( \coth(\cdot) \) is an odd function. The third equality follows from (20). The fifth equality follows from (1). The sixth equality follows from (9). The last equality follows from (26).

Therefore, we get

\[
\coth \left( \frac{i\Omega H}{2} \right) i\Omega = \sigma(i\Omega)(i\Omega)
\]

\[= \sigma. \tag{36} \]

Similarly, the relation in (24) can be derived.

### 4 Uncertainty relation and symplectic eigenvalues of a covariance matrix

Previously, we proved that the following uncertainty relation holds for any \( n \)-mode quantum state that has a finite covariance matrix \( \sigma \):

\[
\sigma + i\Omega \geq 0 . \tag{37}
\]

We now discuss the restriction imposed by the uncertainty relation in (37) on the symplectic eigenvalues of \( \sigma \). Let \( S \) be the symplectic matrix diagonalizing \( \sigma \) as

\[
S\sigma S^T = D = \bigoplus_{j=1}^{n} d_j \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} . \tag{38}
\]

We now prove that (37) implies \( d_j \geq 1, \forall j \). Consider the following chain of inequalities:

\[
\sigma + i\Omega \geq 0 \quad \Rightarrow \quad S(\sigma + i\Omega)S^T \geq 0
\]

\[
\Rightarrow \quad S\sigma S^T + i\Omega S^T \geq 0
\]

\[
\Rightarrow \quad D + i\Omega \geq 0
\]

\[
\Rightarrow \quad \bigoplus_{j=1}^{n} d_j \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + i \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \geq 0
\]

\[
\Rightarrow \quad \bigoplus_{j=1}^{n} \begin{bmatrix} d_j & i \\ -i & d_j \end{bmatrix} \geq 0
\]

\[
\Rightarrow \quad \begin{bmatrix} d_j & i \\ -i & d_j \end{bmatrix} \geq 0, \forall j . \tag{45}
\]

Since the eigenvalues of \( \begin{bmatrix} d_j & i \\ -i & d_j \end{bmatrix} \) are \( d_j + 1 \) and \( d_j - 1 \), it follows from (45) that \( d_j \geq 1, \forall j \).

Thus, any quantum covariance matrix \( \sigma \) (i.e., obeying (37)) has all of its symplectic eigenvalues greater than or equal to one.
5 Purification of a Gaussian state

In this section, we study Gaussian purifications of Gaussian states. We begin by determining the mean vector and covariance matrix for a tensor product of two Gaussian states.

5.1 Tensor product of two Gaussian states

Let $\bar{r}_A$ denote the mean vector and $\sigma_A$ denote the covariance matrix of a Gaussian state $\rho_A$. Let $\bar{r}_B$ denote the mean vector and $\sigma_B$ denote the covariance matrix of a Gaussian state $\rho_B$. Then the mean vector of the tensor product state $\rho_A \otimes \rho_B$ is given by

$$\bar{r}_{AB} \equiv \begin{bmatrix} \bar{r}_A \\ \bar{r}_B \end{bmatrix}. \quad (46)$$

Moreover, the covariance matrix of $\rho_A \otimes \rho_B$ is given by

$$\sigma_{AB} \equiv \sigma_A \oplus \sigma_B = \begin{bmatrix} \sigma_A & 0 \\ 0 & \sigma_B \end{bmatrix}. \quad (47)$$

Similarly, if the mean vector of a Gaussian state is $\begin{bmatrix} \bar{r}_A \\ \bar{r}_B \end{bmatrix}$ and the covariance matrix is $\begin{bmatrix} \sigma_A & 0 \\ 0 & \sigma_B \end{bmatrix}$, then the Gaussian state is a tensor product of two Gaussian states.

5.2 Gaussian purifications of Gaussian states

A thermal state with mean number of photons $\bar{n} \geq 0$ can be expressed in the photon-number basis as follows.

$$\theta(\bar{n}) = \frac{1}{\bar{n} + 1} \sum_{n=0}^{\infty} \left( \frac{\bar{n}}{\bar{n} + 1} \right)^n |n\rangle\langle n|. \quad (48)$$

Alternatively,

$$\theta(\lambda) = \frac{1}{z(\lambda)} \sum_{n=0}^{\infty} \exp(-\lambda(n + 1/2)) |n\rangle\langle n|, \quad (49)$$

where $z(\lambda) = (e^{\lambda/2} - e^{-\lambda/2})^{-1}$ for $\lambda > 0$ (note that $\lambda = \ln(1 + 1/\bar{n})$).

A purification of the thermal state $\theta_A(\bar{n})$ is given by the following two-mode squeezed vacuum (TMS) state:

$$|\psi_{TMS}(\bar{n})\rangle_{AR} = \frac{1}{\sqrt{\bar{n} + 1}} \sum_{n=0}^{\infty} \sqrt{\frac{\bar{n}}{\bar{n} + 1}}^n |n\rangle_A |n\rangle_R, \quad (50)$$

where $R$ is a reference system.
The covariance matrix of the two-mode squeezed vacuum state $|\psi_{TMS}(\bar{n})\rangle_{AR}$ is given by

$$
\begin{bmatrix}
2\bar{n} + 1 & 0 & 2\sqrt{\bar{n}(\bar{n} + 1)} & 0 \\
0 & 2\bar{n} + 1 & 0 & -2\sqrt{\bar{n}(\bar{n} + 1)} \\
2\sqrt{\bar{n}(\bar{n} + 1)} & 0 & 2\bar{n} + 1 & 0 \\
0 & -2\sqrt{\bar{n}(\bar{n} + 1)} & 0 & 2\bar{n} + 1 \\
\end{bmatrix},
$$

which can be written in the following compact form:

$$
\begin{bmatrix}
(2\bar{n} + 1)I & 2\sqrt{\bar{n}(\bar{n} + 1)}\sigma_Z \\
2\sqrt{\bar{n}(\bar{n} + 1)}\sigma_Z & (2\bar{n} + 1)I \\
\end{bmatrix}.
$$

By the Williamson theorem, any $n$-mode Gaussian state $\rho$ can be written as

$$
\rho = \hat{D}_{-\bar{r}}\hat{S}\left(\bigotimes_{j=1}^{n}\theta_{A_j}(\bar{n}_j)\right)\hat{S}^\dagger\hat{D}_{\bar{r}},
$$

where $\hat{S}$ is a unitary generated by a quadratic Hamiltonian. Then a Gaussian purification of $\rho$ is given by

$$
\left[\hat{D}_{-\bar{r}}\hat{S}\right]_{A^n}\left(\bigotimes_{j=1}^{n}|\psi_{TMS}(\bar{n}_j)\rangle_{A_jR_j}\right).
$$

The mean vector of this purification is $[\bar{r}\ 0]$. Moreover, the covariance matrix of this purification is

$$
\begin{bmatrix}
\sigma & S \bigoplus_{j=1}^{n} 2\sqrt{\bar{n}_j(\bar{n}_j + 1)}\sigma_Z \\
\bigoplus_{j=1}^{n} 2\sqrt{\bar{n}_j(\bar{n}_j + 1)}\sigma_Z & \bigoplus_{j=1}^{n}(2\bar{n}_j + 1)I_2 \\
\end{bmatrix}.
$$

One can arrive at this conclusion from the fact that

$$
\sigma = S \left(\bigoplus_{j=1}^{n}(2\bar{n}_j + 1)I_2\right) S^T
$$

and the covariance matrix for $\bigotimes_{j=1}^{n}|\psi_{TMS}(\bar{n}_j)\rangle_{A_jR_j}$ is

$$
\begin{bmatrix}
\bigoplus_{j=1}^{n}(2\bar{n}_j + 1)I_2 & \bigoplus_{j=1}^{n} 2\sqrt{\bar{n}_j(\bar{n}_j + 1)}\sigma_Z \\
\bigoplus_{j=1}^{n} 2\sqrt{\bar{n}_j(\bar{n}_j + 1)}\sigma_Z & \bigoplus_{j=1}^{n}(2\bar{n}_j + 1)I_2 \\
\end{bmatrix}.
$$

We note that the symplectic matrix for the unitary evolution $\hat{S}_{A^n} \otimes I_{R^n}$ is given by

$$
\begin{bmatrix}
S & 0 \\
0 & I \\
\end{bmatrix}.
$$
6 Purity of a quantum state

The purity of a quantum state $\rho$ is defined as $\text{Tr}\{\rho^2\}$. We now show that $\text{Tr}\{\rho^2\} \leq 1$. Consider the following spectral decomposition of the state $\rho$:

$$\rho = \sum_x \lambda_x |\phi_x\rangle\langle \phi_x|.$$  \hfill (59)

Then

$$\text{Tr}\{\rho^2\} = \sum_x \lambda_x^2.$$  \hfill (60)

Since $\lambda_x \leq 1 \Rightarrow \lambda_x^2 \leq 1$ and since $\sum_x \lambda_x = 1 \Rightarrow \sum_x \lambda_x^2 \leq 1$. Therefore, if a state is pure, then $\text{Tr}\{\rho^2\} = 1$.

We now show that if $\text{Tr}\{\rho^2\} = 1$, then the state is pure. Consider that

$$1 = \text{Tr}\{\rho^2\} = \sum_x \lambda_x^2.$$  \hfill (62)

Moreover, $\text{Tr}\{\rho\} = \sum_x \lambda_x = 1 \Rightarrow \text{Tr}\{\rho\}^2 = \sum_{x,y} \lambda_x \lambda_y = 1$.

Consider the following chain of inequalities:

$$0 = \text{Tr}\{\rho^2\} - \text{Tr}\{\rho\}^2$$

$$= \sum_x \lambda_x^2 - \left[ \sum_{x,y} \lambda_x \lambda_y \right]$$

$$= \sum_x \lambda_x^2 - \left[ \sum_x \lambda_x^2 + \sum_{x \neq y} \lambda_x \lambda_y \right]$$

$$= \sum_{x \neq y} \lambda_x \lambda_y.$$ \hfill (66)

Since $\lambda_x, \lambda_y \geq 0$, the only possibility to satisfy (66) is that $\lambda_x = 1$ and $\lambda_y = 0, \forall y \neq x$. Thus, $\text{Tr}\{\rho^2\} = 1$ implies that $\rho$ is a pure state.

6.1 Purity of a Gaussian state

In this section, we calculate the purity for Gaussian states. From the Williamson decomposition of an $n$-mode Gaussian state as defined in (53) and from the fact that the purity is invariant under unitary transformations, we get

$$\text{Tr}\{\rho^2\} = \prod_{j=1}^{n} \text{Tr}\{\theta^2(n_j)\}.$$ \hfill (67)
Consider the following chain of equalities:

\[
\text{Tr}\{\Theta^2(\bar{n}_j)\} = \frac{1}{(\bar{n}_j + 1)^2} \sum_{n=0}^{\infty} \left( \frac{\bar{n}_j}{\bar{n}_j + 1} \right)^{2n}
\]

\[
= \frac{1}{(\bar{n}_j + 1)^2} \frac{1}{1 - (\bar{n}_j/(\bar{n}_j + 1))^2}
\]

\[
= \frac{1}{(\bar{n}_j + 1)^2 - \bar{n}_j^2}
\]

\[
= \frac{1}{2\bar{n}_j + 1}
\]

\[
= \frac{1}{\nu_j},
\]

where \(\nu_j\) denotes the symplectic eigenvalue of \(\Theta(\bar{n}_j)\). The first equality follows from the definition of a thermal state as defined in (48). The second equality follows from the sum of an infinite geometric series.

Therefore,

\[
\text{Tr}\{\rho^2\} = \prod_{j=1}^{n} \frac{1}{\nu_j}
\]

\[
= \sqrt{\prod_{j=1}^{n} \frac{1}{\nu_j^2}}
\]

\[
= \frac{1}{\sqrt{\prod_{j=1}^{n} \nu_j^2}}
\]

\[
= \frac{1}{\text{Det}(\sigma)}.
\]

The last equality follows from (26) and from the fact that for any symplectic matrix \(S\), \(\text{Det}(S) = 1\).

Therefore, the purity of a Gaussian state is

\[
\text{Tr}\{\rho^2\} = \frac{1}{\sqrt{\text{Det}(\sigma)}},
\]

which implies that a Gaussian state is pure if and only if \(\text{Det}(\sigma) = 1\). Since \(\nu_j \geq 1\), an equivalent condition for the purity of a Gaussian state is that all symplectic eigenvalues are equal to one.

### 7 Entropy of a Gaussian state

In this section, we find an expression for the von Neumann entropy of a Gaussian state.

The von Neumann entropy of a quantum state \(\rho\) is defined as

\[
S(\rho) \equiv -\text{Tr}\{\rho \ln \rho\}.
\]
We begin by expressing a thermal state with the mean photon number $\bar{n}$ in the following form:

$$\theta(\bar{n}) = \frac{1}{\bar{n} + 1} \sum_{n=0}^{\infty} \left( \frac{\bar{n}}{\bar{n} + 1} \right)^n \langle n \rangle \langle n |$$

(79)

$$= \frac{1}{\bar{n} + 1} \left( \frac{\bar{n}}{\bar{n} + 1} \right)^{\hat{n}}.$$  

(80)

Consider the following chain of equalities:

$$- \text{Tr} \{ \theta(\bar{n}) \ln \theta(\bar{n}) \} = - \text{Tr} \left\{ \theta(\bar{n}) \ln \left( \frac{1}{\bar{n} + 1} \right) \right\}$$

(81)

$$= - \text{Tr} \left\{ \theta(\bar{n}) \ln \left( \frac{\bar{n}}{\bar{n} + 1} \right) \right\} - \text{Tr} \left\{ \theta(\bar{n}) \hat{n} \ln \left( \frac{\bar{n}}{\bar{n} + 1} \right) \right\}$$

(82)

$$= \ln(\bar{n} + 1) - \text{Tr} \theta(\bar{n}) \hat{n}$$

(83)

$$= \ln(\bar{n} + 1) - \text{Tr} \theta(\bar{n}) \hat{n} \ln \bar{n}$$

(84)

$$\equiv g(\bar{n}).$$

(85)

From unitary invariance and additivity of the von Neumann entropy, we get

$$S(\rho) = S \left( \bigotimes_{j=1}^{n} \theta(\bar{n}_j) \right),$$

(87)

where $\rho$ is an $n$-mode Gaussian state as defined in (53). Therefore,

$$S(\rho) = \sum_{j=1}^{n} S(\theta(\bar{n}_j))$$

(88)

$$= \sum_{j=1}^{n} g(\bar{n}_j)$$

(89)

We now derive an alternative formula for the von Neumann entropy of faithful Gaussian states. Let

$$\rho = \frac{1}{\sqrt{\text{Det}[(\sigma + i\Omega)/2]}} \exp \left( -\frac{1}{2} (\hat{r} - \bar{r})^T H (\hat{r} - \bar{r}) \right)$$

(90)

$$= \hat{D}_{\bar{r}} \left[ \exp \left( -\frac{1}{2} \hat{r}^T H \hat{r} \right) \right] \hat{D}_{\bar{r}}$$

(91)

and let

$$\rho_0 = \frac{\exp \left( -\frac{1}{2} \hat{r}^T H \hat{r} \right)}{\sqrt{\text{Det}[(\sigma + i\Omega)/2]}}.$$  

(92)
Then from unitary invariance of the von Neumann entropy, we get

\[ S(\rho) = S(\rho_0) \]

\[ = - \text{Tr}\{\rho_0 \ln \rho_0\} \]

\[ = - \text{Tr}\{\rho_0 \ln \frac{\exp(-\frac{1}{2} \hat{r}^T H \hat{r})}{\sqrt{\text{Det}[(\sigma + i\Omega)/2]}}\} \]

\[ = - \text{Tr}\{\rho_0 \ln \frac{1}{\sqrt{\text{Det}[(\sigma + i\Omega)/2]}}\} - \text{Tr}\{\rho_0 \ln \exp(-\frac{1}{2} \hat{r}^T H \hat{r})\} \]

\[ = \frac{1}{2} \ln \text{Det}[(\sigma + i\Omega)/2] + \frac{1}{2} \text{Tr}\{\rho_0 \hat{r}^T H \hat{r}\}. \]

We now focus on the second term of the aforementioned equation.

\[ \text{Tr}\{\rho_0 \hat{r}^T H \hat{r}\} = \text{Tr}\{\rho_0 \sum_{j,k} \hat{r}_j H_{j,k} \hat{r}_k\} \]

\[ = \sum_{j,k} H_{j,k} \text{Tr}\{\rho_0 \hat{r}_j \hat{r}_k\} \]

\[ = \frac{1}{2} \sum_{j,k} H_{j,k} \text{Tr}\{\rho_0 (\{\hat{r}_j, \hat{r}_k\} + [\hat{r}_j, \hat{r}_k])\} \]

\[ = \frac{1}{2} \sum_{j,k} H_{j,k} (\sigma_{j,k} + i\Omega_{j,k}) \]

\[ = \frac{1}{2} \sum_{j,k} H_{j,k} \sigma_{j,k} - \frac{i}{2} \sum_{j,k} H_{j,k} \Omega_{k,j} \]

\[ = \frac{1}{2} \text{Tr}\{H\sigma\} - \frac{i}{2} \text{Tr}\{H\Omega\} \]

\[ = \frac{1}{2} \text{Tr}\{H\sigma\}, \]

where we used the fact that \(\text{Tr}\{H\Omega\} = 0\), which holds because \(H\) is symmetric and \(\Omega\) is antisymmetric.

Therefore,

\[ S(\rho) = \frac{1}{2} \ln \text{Det}[(\sigma + i\Omega)/2] + \frac{1}{4} \text{Tr}\{H\sigma\}. \]

Moreover, from [24] it follows that

\[ S(\rho) = \frac{1}{2} \ln \text{Det}[(\sigma + i\Omega)/2] + \frac{1}{2} \text{Tr}\{\text{arccoth}(i\Omega\sigma)i\Omega\sigma\}. \]

This latter expression is valid for pure Gaussian states, with the expression \(\text{Tr}\{\text{arccoth}(i\Omega\sigma)i\Omega\sigma\}\) understood in a limiting sense.

References