**PHYS 7895: Gaussian Quantum Information**

# Lecture 5

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## 1 Overview

In this lecture, we work mostly with single-mode bosonic quantum systems. We first formally define photon-number states (also known as Fock states). We then introduce the annihilation and creation operators in Section 2.2 and the quadrature operators in Section 2.3. Then we study the existence of normalized eigenvectors for the introduced operators in Section 2.4. We end this lecture by developing the background for multiple-mode systems in Section 3.

## 2 Single-mode systems

A mode, informally, refers to a well defined degree of freedom of the system. An example of a single-mode bosonic quantum system is a photonic degree of freedom with a well defined polarization or frequency. Mathematically, a bosonic mode is described by a separable Hilbert space (that is, a Hilbert space that admits a countable orthonormal basis) equipped with canonical operators.

### 2.1 Photon-number states

Recall the Kronecker functions discussed earlier in Lecture 2 in the context of $l^2(\mathbb{N})$ space. Analogous to this, let us define the set $\{|n\rangle\}_{n=0}^{\infty}$ of photon-number states, which form a countable basis set for the separable Hilbert state. In second quantization, photon-number states correspond to the number of photons in a single mode of a bosonic system, that is, the number of photons in the system with particular frequency and a particular polarization. Since the photon-number states form a basis set for the separable Hilbert state, any state can be represented in terms of these states.

### 2.2 Annhilation and creation operators

Now, let us first define the **annihilation operator** by its action on the photon-number basis:

$$\hat{a}\,|n\rangle = \sqrt{n}\,|n-1\rangle \quad \forall n \geq 1, \tag{1}$$

$$\hat{a}\,|0\rangle = 0. \tag{2}$$

From this, we deduce that matrix elements of the annihilation operator in the photon number basis are given as

$$\langle m| (\hat{a} |n\rangle) = \sqrt{n}\ \delta_{m,n-1} \quad \forall n \geq 1. \tag{3}$$

$$\langle m| (\hat{a} |n\rangle) = 0,\ \text{for } n = 0. \tag{4}$$

**Proposition 1.** *The annihilation operator $\hat{a}$ is an unbounded operator.*

*Proof.* The operator norm is defined as

$$\|\hat{a}\| = \sup_{\|\phi\|=\|\psi\|=1} |\langle\phi| \hat{a} |\psi\rangle| \tag{5}$$

Choose $|\phi\rangle = |n-1\rangle$, and $|\psi\rangle = |n\rangle$. Then,

$$|\langle\phi| \hat{a} |\psi\rangle| = \sqrt{n}. \tag{6}$$

By taking the limit $n \to \infty$, we find that

$$\sup_{\|\phi\|=\|\psi\|=1} |\langle\phi| \hat{a} |\psi\rangle| \geq \lim_{n\to\infty} \sqrt{n} = \infty. \tag{7}$$

So we conclude that $\sup_{\|\phi\|=\|\psi\|=1} |\langle\phi| \hat{a} |\psi\rangle| = \infty$. □

Let us now define the **creation operator** $\hat{a}^\dagger$ as the adjoint of the annihilation operator $\hat{a}$. We recover the action of the creation operator on $|n\rangle$, from the properties that we have established for the annhilation operator. Consider that

$$\langle m| \left(\hat{a}^\dagger |n\rangle\right) = (\hat{a} |m\rangle)^\dagger |n\rangle = \sqrt{m}\ \delta_{m-1,n}. \tag{8}$$

Set $m = n+1$. Then, $\langle n+1|\hat{a}^\dagger n\rangle = \sqrt{n+1}$. Since $\langle m|\hat{a}^\dagger n\rangle = 0$ for $m \neq n+1$, this implies

$$\hat{a}^\dagger |n\rangle = \sqrt{n+1}\, |n+1\rangle \qquad \forall n \geq 0. \tag{9}$$

We can prove that the creation operator is also unbounded by following an argument similar to the one for the annihilation operator.

Now, we obtain the canonical commutation relation (CCR) for the annihilation and creation operators:

$$\left[\hat{a}, \hat{a}^\dagger\right] = \hat{I}, \tag{10}$$

where the $\hat{I}$ is the identity operator for the separable Hilbert space. Consider the action of $\left[\hat{a}, \hat{a}^\dagger\right]$ on an arbitrary number state $|n\rangle$:

$$\left[\hat{a}, \hat{a}^\dagger\right] |n\rangle = \left(\hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\right) |n\rangle = |n\rangle \quad \forall n \geq 0, \tag{11}$$

where we have skipped some algebraic steps. Since this holds for an orthonormal basis, we conclude that $\left[\hat{a}, \hat{a}^\dagger\right] = \hat{I}$. Similarly, we can obtain that $\left[\hat{a}^\dagger, \hat{a}\right] = -\hat{I}$, $\left[\hat{a}^\dagger, \hat{a}^\dagger\right] = 0$ and $[\hat{a}, \hat{a}] = 0$. We can then capture these CCR in a matrix as

$$\begin{bmatrix} \left[\hat{a}, \hat{a}^\dagger\right] & [\hat{a}, \hat{a}] \\ \left[\hat{a}^\dagger, \hat{a}^\dagger\right] & \left[\hat{a}^\dagger, \hat{a}\right] \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \otimes \hat{I} = \sigma_z \otimes \hat{I},$$

It is easy to obtain the following:

$$\hat{a}^\dagger \hat{a} |n\rangle = n |n\rangle . \tag{12}$$

So, the photon-number states $|n\rangle$ are eigenstates of $\hat{a}^\dagger \hat{a} = \hat{n}$ with eigenvalue $n$. Therefore, we can write

$$\hat{n} = \sum_{n=0}^{\infty} n |n\rangle\langle n| . \tag{13}$$

The operator $\hat{n}$ is known as the photon-number operator.

## 2.3 Position and Quadrature operators

Let us now define the position and momentum quadrature operators as

$$\hat{x} = \frac{\hat{a} + \hat{a}^\dagger}{\sqrt{2}}, \qquad \hat{p} = \frac{\hat{a} - \hat{a}^\dagger}{\sqrt{2}i}. \tag{14}$$

By definition, these are Hermitian operators and can be compactly written as

$$\begin{bmatrix} \hat{x} \\ \hat{p} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix} \begin{bmatrix} \hat{a} \\ \hat{a}^\dagger \end{bmatrix}. \tag{15}$$

By rearrangement, we obtain the following:

$$\begin{bmatrix} \hat{a} \\ \hat{a}^\dagger \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{p} \end{bmatrix}. \tag{16}$$

The quadrature operators $\hat{x}$ and $\hat{p}$ are unbounded since $\hat{a}$ and $\hat{a}^\dagger$ are unbounded. From the commutation relations of $\hat{a}$ and $\hat{a}^\dagger$, we can work out the commutation relations of $\hat{x}$ and $\hat{p}$. The CCR of the quadrature operators can then be embedded in a matrix as follows:

$$\begin{bmatrix} [\hat{x},\hat{x}] & [\hat{x},\hat{p}] \\ [\hat{p},\hat{x}] & [\hat{p},\hat{p}] \end{bmatrix} = i \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \otimes \hat{I}. \tag{17}$$

## 2.4 Eigenvectors of $\hat{x}, \hat{p}, \hat{a}, \hat{a}^\dagger$

In this section, we show that $\hat{x}$, $\hat{p}$, and $\hat{a}^\dagger$ do not have normalized eigenvectors, and that the coherent states (which we define later) are the normalized eigenvectors of the annihilation operator.

**Proposition 2.** *The quadrature operators $\hat{x}$ and $\hat{p}$ do not have normalized eigenvectors.*

*Proof.* Suppose that $|\psi\rangle$ is an eigenvector of $\hat{x}$. That is,

$$\hat{x} |\psi\rangle = \lambda |\psi\rangle , \tag{18}$$

where $\lambda \in \mathbb{R}$. Now, we know that $[\hat{x}, \hat{p}] = i\hat{I}$. Next, consider that

$$\langle\psi| [\hat{x},\hat{p}] |\psi\rangle = \langle\psi| \hat{x}\hat{p} - \hat{p}\hat{x} |\psi\rangle \tag{19}$$

$$= \langle\psi| \lambda\hat{p} - \hat{p}\lambda |\psi\rangle = 0. \tag{20}$$

However, $\langle\psi| i\hat{I} |\psi\rangle = i$. This leads to a contradiction, and implies that $\hat{x}$ cannot have a normalized eigenvector. $\square$

Following a similar argument, we can prove that $\hat{p}$ cannot have a normalized eigenvectors.

**Proposition 3.** *The creation operator $\hat{a}^\dagger$ does not have a normalized eigenvector.*

*Proof.* Suppose that there exists a normalized eigenvector $|\psi\rangle$ such that

$$\hat{a}^\dagger |\psi\rangle = \mu |\psi\rangle \quad \forall \mu \in \mathbb{C}. \tag{21}$$

We can write $|\psi\rangle = \sum_{n=0}^{\infty} c_n |n\rangle$ for $c_n \in \mathbb{C}$ such that $\sum_{n=0}^{\infty} |c_n|^2 = 1$. Then, it follows from (21) that

$$\hat{a}^\dagger |\psi\rangle = \sum_{n=0}^{\infty} c_n \hat{a}^\dagger |n\rangle = \sum_{n=0}^{\infty} c_n \sqrt{n+1}\, |n+1\rangle \tag{22}$$

$$= \sum_{n=0}^{\infty} c_n \mu |n\rangle . \tag{23}$$

This implies that $c_0\mu = 0$, $c_0 = c_1\mu$, $c_1\sqrt{2} = c_2\mu$, and so on. If $\mu \neq 0$, then $c_0 = 0$ and this implies $c_n = 0$, where $n \in \mathbb{N}$. If $\mu = 0$, then also $c_0 = 0$ and this implies $c_n = 0$, where $n \in \mathbb{N}$. Therefore, the creation operator $\hat{a}^\dagger$ does not have a normalized eigenvector. □

Interestingly, the annihilation operator $\hat{a}$ has normalized eigenstates, which are called **coherent states**. Each of the coherent states are parametrized by $\alpha \in \mathbb{C}$. That is, $\hat{a}|\alpha\rangle = \alpha |\alpha\rangle$.

**Proposition 4.** *The annihilation operator $\hat{a}$ has coherent states $|\alpha\rangle$ as its normalized eignevectors.*

*Proof.* Let us suppose that a coherent state $|\alpha\rangle$ is an eigenstate of $\hat{a}$. Expanding $|\alpha\rangle$ in terms of the number basis, we obtain

$$|\alpha\rangle = \sum_{n=0}^{\infty} c_n |n\rangle \quad \text{for } c_n \in \mathbb{C}. \tag{24}$$

Now apply $\hat{a}$ to $|\alpha\rangle$:

$$\hat{a}|\alpha\rangle = \hat{a} \sum_{n=0}^{\infty} c_n |n\rangle \tag{25}$$

$$= \sum_{n=0}^{\infty} c_n \hat{a} |n\rangle \tag{26}$$

$$= \sum_{n=1}^{\infty} c_n \sqrt{n}\, |n-1\rangle . \tag{27}$$

To be an eigenstate, $|\alpha\rangle$ should satisfy the following relation:

$$\hat{a}|\alpha\rangle = \alpha |\alpha\rangle \tag{28}$$

$$\sum_{n=1}^{\infty} c_n \sqrt{n}\, |n-1\rangle = \sum_{n=0}^{\infty} \alpha\ c_n |n\rangle . \tag{29}$$

Equating coefficients term by term gives the following recursion relation:

$$c_n \sqrt{n} = \alpha\ c_{n-1}, \tag{30}$$

which gives us the following:

$$c_n = \frac{\alpha^n}{\sqrt{n!}} c_0. \tag{31}$$

Then,

$$|\alpha\rangle = c_0 \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle . \tag{32}$$

From the normalization condition, we can fix the value of $c_0$. We thus have,

$$\begin{align} 1 &= \langle \alpha | \alpha \rangle \tag{33} \\ &= |c_0|^2 \sum_{n,n'=0}^{\infty} \frac{\alpha^{*n} \alpha^{n'}}{\sqrt{n! n'!}} \langle n | n' \rangle \tag{34} \\ &= |c_0|^2 \sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{n!} \tag{35} \\ &= |c_0|^2 e^{|\alpha|^2}. \tag{36} \end{align}$$

This implies

$$|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle , \tag{37}$$

concluding the proof. □

# 3 Multiple modes

Earlier in this lecture, we have been concentrating on single mode systems. Now, we move on to study multiple-mode bosonic Hilbert spaces. By tensoring together several separable Hilbert spaces, each corresponding to a bosonic mode, we get a multiple-mode bosonic Hilbert space. Each mode $j$ is equipped with canonical operators $\hat{x}_j$ and $\hat{p}_j$ for $j \in \{1, \ldots m\}$, where $m$ is the number of modes. If $j \neq k$, then $[\hat{x}_j, \hat{p}_k] = 0$, since these operators are acting on different Hilbert spaces. Now, we can encode these canonical commutation relations as

$$[\hat{x}_j, \hat{p}_k] = i\delta_{j,k}\hat{I}. \tag{38}$$

To write the relations compactly, we define the vector of the canonical operators as

$$\hat{r} = \begin{bmatrix} \hat{x}_1 \\ \hat{p}_1 \\ \vdots \\ \hat{x}_n \\ \hat{p}_n \end{bmatrix} \tag{39}$$

Then, the CCR can be encoded in the following matrix:

$$\left[\hat{r}, \hat{r}^\dagger\right] = \begin{bmatrix} [\hat{x}_1, \hat{x}_1] & [\hat{x}_1, \hat{p}_1] & \cdots \\ [\hat{p}_1, \hat{x}_1] & [\hat{p}_1, \hat{p}_1] & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix} = i\Omega, \tag{40}$$

where $\Omega = \bigoplus_{j=1}^{n} \Omega_1 = I_n \otimes \Omega_1$ with

$$\Omega_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \tag{41}$$

$\Omega$ is a special matrix, called the symplectic form, which realizes a symplectic inner product via $x^T \Omega\, y$. Some properties of the symplectic form are the following:

- $\Omega^T = -\Omega$
- $\Omega^T \Omega = -\Omega^2 = I_{2n}$. That is, $\Omega$ is an orthogonal matrix.
- Commutator matrix $i\Omega$ is involutory, that is, $(i\Omega)^2 = I$.

One can also use a different order for vectors of canonical operators as

$$\hat{s} = \begin{bmatrix} \hat{x}_1 \\ \vdots \\ \hat{x}_n \\ \hat{p}_1 \\ \vdots \\ \hat{p}_n \end{bmatrix} \tag{42}$$

In this case,

$$\left[\hat{s}, \hat{s}^\dagger\right] = iJ, \tag{43}$$

where

$$J = \begin{bmatrix} 0_n & I_n \\ -I_n & 0_n \end{bmatrix} = \Omega_1 \otimes I_n. \tag{44}$$

Another convention often used in the literature is the following:

$$\underline{\hat{a}} = \begin{bmatrix} \hat{a}_1 \\ \hat{a}_1^\dagger \\ \vdots \\ \hat{a}_n \\ \hat{a}_n^\dagger \end{bmatrix}. \tag{45}$$

Then, $\underline{\hat{a}} = U\hat{r}$, where the unitary $U$ is defined as

$$U = \bigoplus_{j=1}^{n} u = I_n \otimes u, \tag{46}$$

with

$$u = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix}. \tag{47}$$

Then,

$$
\begin{aligned}
\left[\underline{\hat{a}}, \underline{\hat{a}^\dagger}\right] &= \left[U\hat{r}, \hat{r}^\dagger U^\dagger\right] && (48)\\
&= U\left[\hat{r}, \hat{r}^\dagger\right]U^\dagger && (49)\\
&= Ui\left(I_n \otimes \Omega_1\right)U^\dagger && (50)\\
&= i\left(I_n \otimes u\right)\left(I_n \otimes \Omega_1\right)\left(I_n \otimes u^\dagger\right) && (51)\\
&= iI_n \otimes u\Omega_1 u^\dagger && (52)\\
&= I_n \otimes \sigma_z && (53)\\
&= \bigoplus_{i=1}^{n} \sigma_z. && (54)
\end{aligned}
$$

This implies,

$$
\left[\underline{\hat{a}}, \underline{\hat{a}^\dagger}\right] = I_n \otimes \sigma_z. \tag{55}
$$

One can also define another order

$$
\overline{\hat{a}} = \begin{bmatrix} \hat{a}_1 \\ \hat{a}_1^\dagger \\ \vdots \\ \hat{a}_n \\ \hat{a}_n^\dagger \end{bmatrix}. \tag{56}
$$

However, we do not go into the details of this convention.