Lecture 14

wave equation - one of the most ~~importatet~~ equations in all of physics:

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$$

- this governs the travel of any kind of wave, not just a single sine wave. (of course $y_m \sin(kx \pm \omega t)$ is a solution)

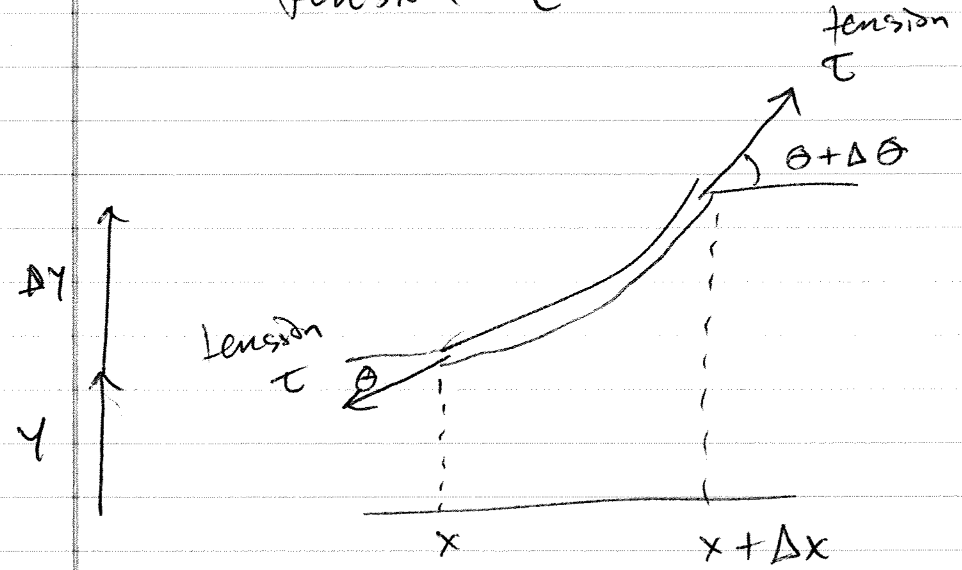
- It is a linear differential equation & is a consequence of the Newton 2nd law.

- relates changes in space to changes in time

②

can derive it quickly

Consider a small part of the string w/ mass dm & tension τ



concerned only w/ up & down motion of this piece of string

total force in y direction is

$$F_y = \underbrace{-\tau \sin\theta}_{\text{downward force on left}} + \underbrace{\tau \sin(\theta+\Delta\theta)}_{\text{upward force on right}}$$

small angle approximation gives

$$F_y = -\tau\theta + \tau(\theta+\Delta\theta)$$

$$= \tau\Delta\Theta$$

Now figure out ma for F=ma

mass of string element is

$$dm = \underset{\substack{\uparrow\\ \text{linear}\\ \text{mass}\\ \text{density}}}{\mu}\quad \underset{\substack{\uparrow\\ \text{length}\\ \text{of}\\ \text{string element}}}{\Delta x}$$

acceleration is $a_y = \frac{\partial^2 y}{\partial t^2}$

$F = ma$

$$\Rightarrow \quad \tau\Delta\Theta = (\mu\Delta x)\frac{\partial^2 y}{\partial t^2}$$

$$\Rightarrow \quad \Delta\Theta/\Delta x = \mu/\tau \frac{\partial^2 y}{\partial t^2}$$

Consider that $\frac{\Delta y}{\Delta x} = \tan\Theta$

Since string element is small,
think of this as

$$\underbrace{\frac{\partial y}{\partial x}}_{\substack{\text{partial deriv.,}\\ \text{b/c snapshot @}\\ \text{some time}}} \simeq \tan\theta$$

we are interested in $\frac{\Delta\theta}{\Delta x}$,
so think of θ as function of x,
+ take partial wrt x to get

$$\frac{\partial^2 y}{\partial x^2} = \frac{\partial \tan\theta(x)}{\partial x} = \frac{1}{\cos^2\theta}\frac{\partial\theta}{\partial x}$$

for small angle, $\cos^2\theta \approx 1$

$$\Rightarrow \frac{\partial^2 y}{\partial x^2} = \frac{\partial\theta}{\partial x}$$

$$\Rightarrow \frac{\partial^2 y}{\partial x^2} = \mu/\tau\, \frac{\partial^2 y}{\partial t^2}$$

$$= 1/v^2\, \frac{\partial^2 y}{\partial t^2}$$

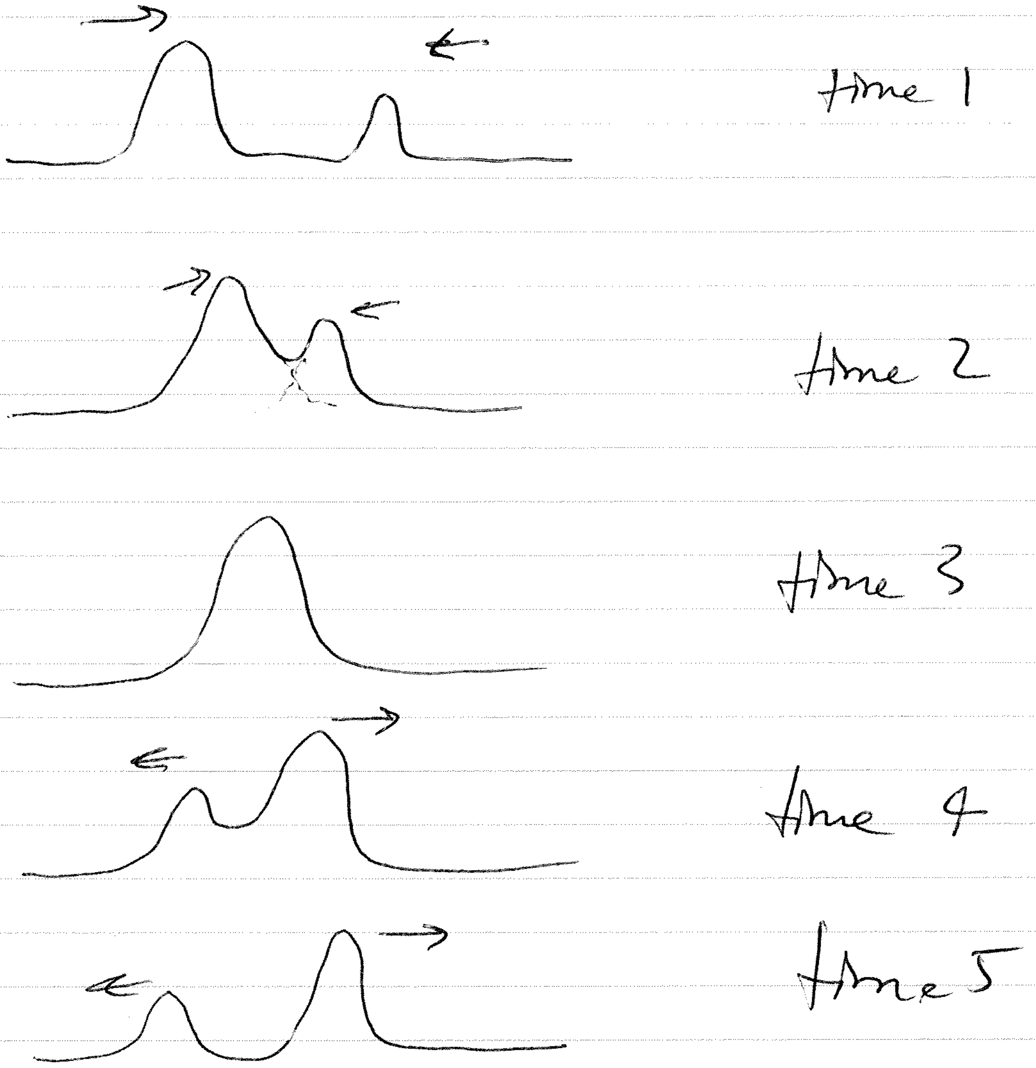
principle of superposition for waves

- Since wave equation is a linear diff. eq., ~~a solution~~ the sum of two different solutions is also a solution.
- So if $y_1(x,t)$ & $y_2(x,t)$ represent waves, then

$$y(x,t) = y_1(x,t) + y_2(x,t)$$

is another wave.

- For a string, this means that two waves traveling along it will just overlap



@ each time instance,
wave is algebraic sum of
two waves

Also, overlapping waves do not
alter the travel of each other.

Interference of waves

- Suppose we send two sine waves of same wavelength + amplitude along a stretched string
- What is the resultant wave?

- depends on whether wave are in phase or out of phase
- combination of waves is called interference (waves interfere w/ each other)

Example:

Suppose $y_1(x,t) = y_m \sin(kx - \omega t)$

& $y_2(x,t) = y_m \sin(kx - \omega t + \phi)$

these are out of phase by ϕ

then resultant wave is

$y_1(x,t) + y_2(x,t)$

$= y_m \sin(kx - \omega t) + y_m \sin(kx - \omega t + \phi)$

use $\sin\alpha + \sin\beta$

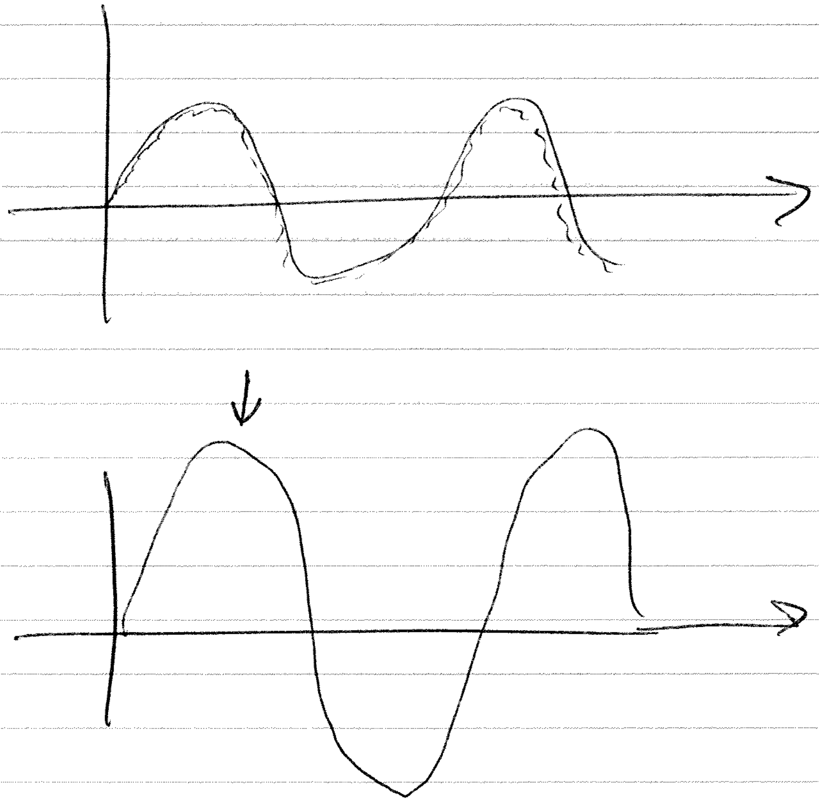
$= 2\sin\left(\frac{1}{2}(\alpha+\beta)\right)\cos\left(\frac{1}{2}(\alpha-\beta)\right)$

$$\Rightarrow \; = \underbrace{2y_m \cos\left(\tfrac{1}{2}\phi\right)}_{\text{this is the new amplitude}} \; \underbrace{\sin\left(kx - \omega t + \tfrac{1}{2}\phi\right)}_{\text{oscillating term}}$$

If $\phi = 0$, then waves are exactly in phase + this simplifies to

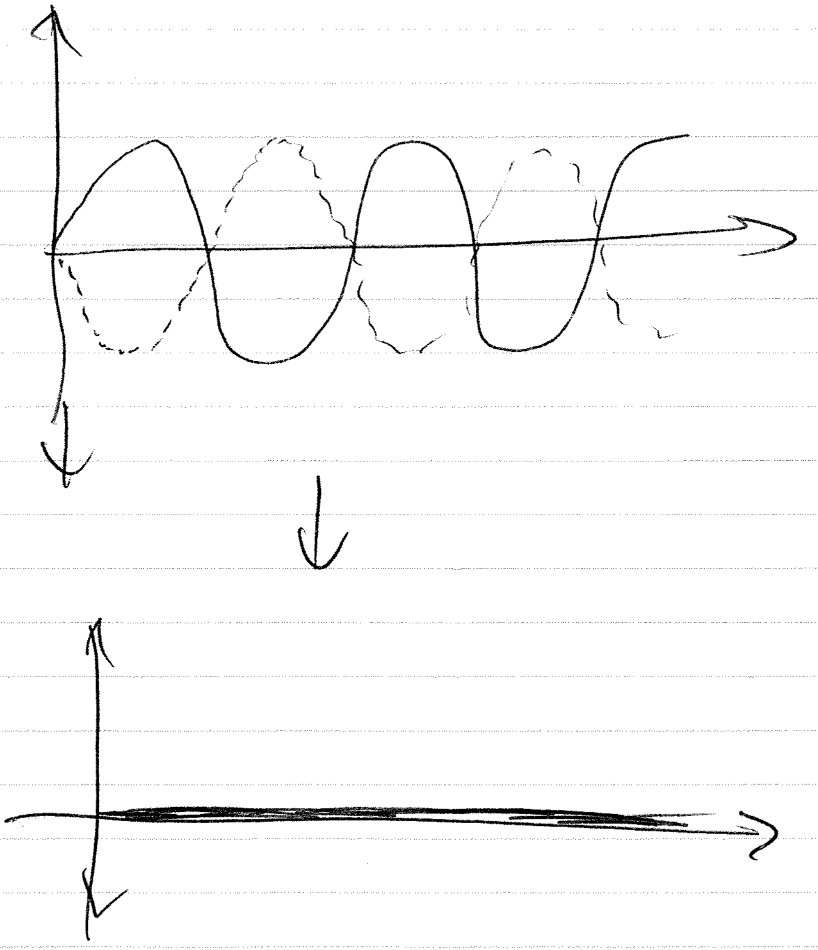
$$2y_m \sin(kx - \omega t)$$

picture is



fully constructive interference

- If $\phi = \pi$, then waves are *exactly* out of phase & $\cos(\frac{1}{2}\phi) = 0$. Then resultant wave is 0.
- Picture is

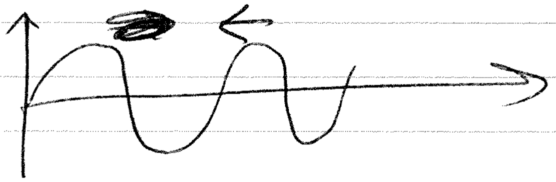


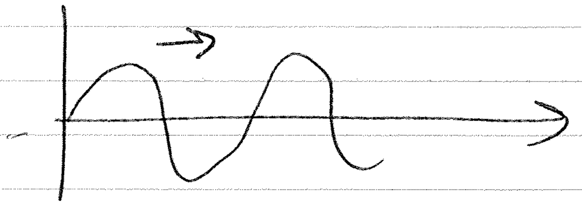
fully destructive interference

for other values of ϕ,
interference is <u>intermediate</u>
(neither fully constructive or
destructive)

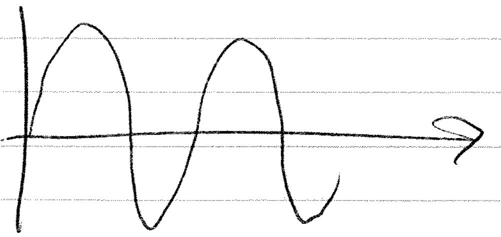
Standing waves

What if sine waves travel
in opposite directions?

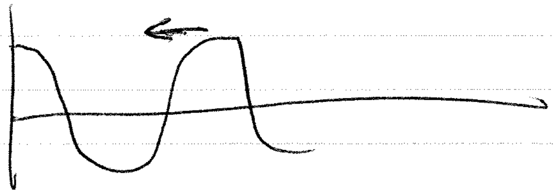


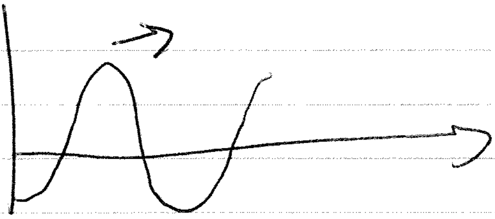


sum: is



time 2





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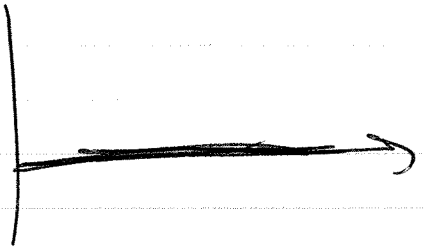

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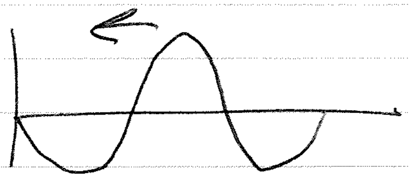
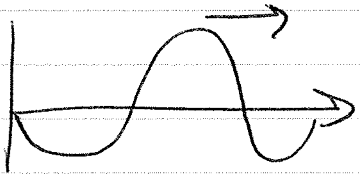
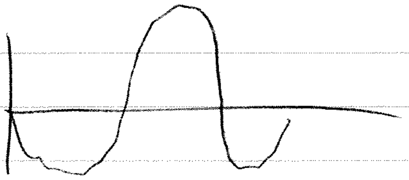
time 3





sum is



to analyze, use

$$y_1(x,t) = y_m \sin(kx - \omega t)$$ traveling to the right

$$y_2(x,t) = y_m \sin(kx + \omega t)$$ traveling to left

Then

$$y_1(x,t) + y_2(x,t) = y_m \sin(kx - \omega t) + y_m \sin(kx + \omega t)$$

$$= \underbrace{2y_m \sin(kx)\cos(\omega t)}_{\text{this is not a traveling wave!}}$$

doesn't have the form

$$\sin(kx - \omega t + \phi)$$

this is a <u>standing wave</u>, where amplitude varies w/ position