Lecture 13

wave speed on a stretched string

recall that wave speed, v is given by

$$v = \frac{\omega}{k} = \frac{\lambda}{T} = \lambda f$$

- This speed is set by the properties of the medium through which the wave travels.

- It is a function of the tension τ in the string & its linear mass density.

- [illegible] wave doesn't travel along string unless there is tension, which is the magnitude of the forces on each end needed to make it taut.

- linear mass density μ of string is mass per unit length

units of tension are units of force τ – $\dfrac{\text{mass} \cdot \text{length}}{\text{time}^2}$

units of μ are $\dfrac{\text{mass}}{\text{length}}$

- any guess as to what speed v should be as a function of τ & μ?

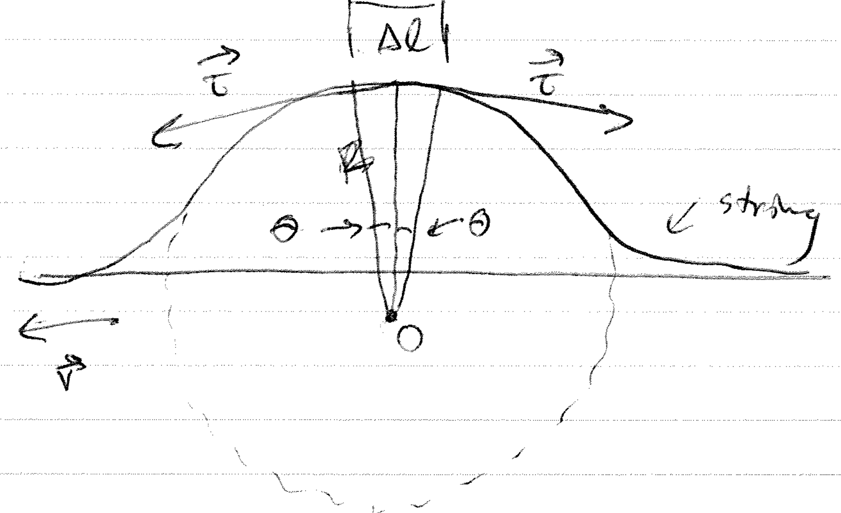
- can use dimensional analysis to guess

$$v = C\sqrt{\frac{\tau}{\mu}}$$

where C is a dimensionless constant

- To really figure it out, we need to start from basic physical principles + derive it (we'll see that $C=1$)
- starting point is the Newton 2nd law

Consider a single symmetrical pulse:



wave is going from right to left

linear mass density $\mu = \frac{m}{\ell}$

- In this analysis, we are considering a small arc of a circle of radius R & an arc for the angle 2θ

- the force $\vec{\tau}$ from tension in the string pulls it to the left & to the right
- horizontal components of forces cancel, but vertical components add to make a downward restoring force $\vec{F}$
- magnitude is

$$F = 2(\tau \sin\theta)$$

$$\approx 2\tau\theta$$

$$= \frac{\tau \Delta l}{R}$$

- mass of the element is

$$\Delta m = \mu \Delta l$$

- In this moment, string element Δl is moving in the arc of a circle, & so it has a centripetal acceleration given by

$$a = \frac{v^2}{R}$$

- Newton 2nd law is

$$F = m a$$

$$\Rightarrow \quad \frac{\tau \Delta l}{R} = \mu \Delta l \frac{v^2}{R}$$

$$\Rightarrow \quad \tau = \mu v^2$$

$$\Rightarrow \quad \boxed{v = \sqrt{\frac{\tau}{\mu}}}$$

- Velocity of wave (wave speed) depends only on tension T & linear mass density μ

- it does not depend on the frequency of the wave.

- frequency of the wave is fixed by the source that is generating it.

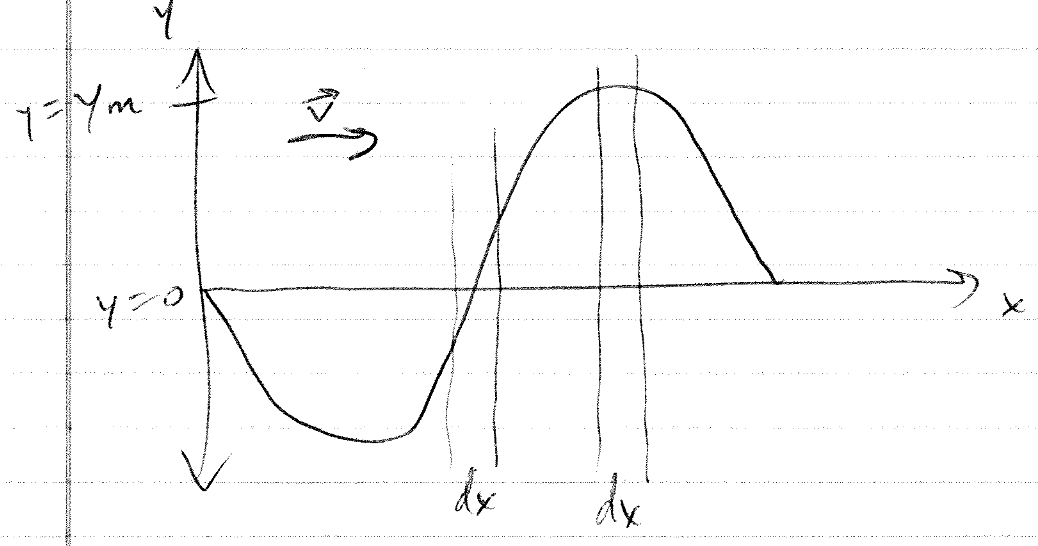
Energy and power of a wave traveling along a string

- A wave on a stretched string generates energy.

- It transports energy as both kinetic & elastic potential energy

1st consider kinetic energy,

Take a snapshot of wave in time



- String element of mass dm oscillates transversely in SHM as wave passes through it

- @ $y=0$, transverse velocity is @ a maximum, & @ $y = y_m$, transverse velocity is minimum

- so kinetic energy @ $y=0$ is @ maximum + @ $y=y_m$ is minimum

- As wave travels along string, forces due to tension in rope do work to transport energy along string.

Rate of Energy Transmission

Kinetic energy dK associated w/ string element of mass dm is

$$dK = \frac{1}{2}\, dm\, u^2$$

where u is transverse speed.

To find u, take derivative of

$$y(x,t) = y_m \sin(kx - \omega t)$$

wrt time t.

$$\frac{\partial y(x,t)}{\partial t} = -\omega y_m \cos(kx-\omega t)$$

$$dm = \mu\, dx$$

$$\Rightarrow \quad dK = \frac{1}{2}\mu\, dx\, (-\omega y_m)^2 \cos^2(kx-\omega t)$$

$$\Rightarrow$$

$$\frac{dK}{dt} = \frac{1}{2}\mu \frac{dx}{dt}\, \omega^2 y_m^2 \cos^2(kx-\omega t)$$

$$= \frac{1}{2}\mu\, v\, \omega^2 y_m^2 \cos^2(kx-\omega t)$$

wave speed

average kinetic energy rate is then

$$\left(\frac{dK}{dt}\right)_{avg} = \frac{1}{4}\mu v^2 \omega^2 y_m^2$$

b|c $\left(\cos^2(kx-\omega t)\right)_{avg} = \frac{1}{2}$

average is over integer # of wavelengths

- Elastic potential energy is also associated w/ wave propagation due to stretching of string as wave goes by

- a string element of length dx has its length increase + decrease

- @ $y=0$, string element has maximum stretch + so has maximum elastic potential energy here.

(no violation of conservation of energy because wave is transporting energy along string.)

— It is possible to show that average potential energy is same as that for kinetic energy, so that

$$\left(\frac{dU}{dt}\right)_{avg} = \left(\frac{dK}{dt}\right)_{avg}$$

⇒ average power is (which is avg. rate @ which both kinds of energy are transmitted by wave)

$$\boxed{\begin{aligned} P_{avg} &= 2\left(\frac{dK}{dt}\right)_{avg} \\ &= \frac{1}{2}\mu v \omega^2 y_m^2 \end{aligned}}$$

μ & v depend on material & tension of string.

ω & y_m depend on process that generates wave.