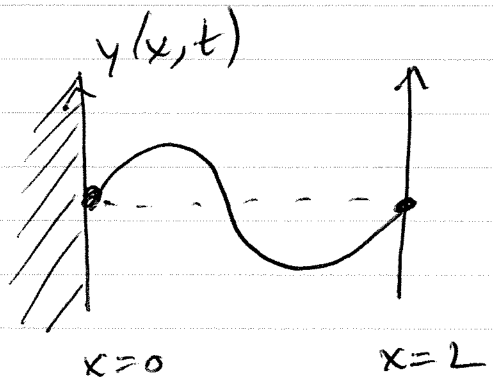
# Lecture 26

Suppose that there is a string w/ its ~~two~~ ends attached to seperate walls




Suppose displacement y is small & slop $\frac{\partial y}{\partial x}$ is small as well.

Under these assumptions, displacement y(x,t) satisfies the one-dimensional wave equation:

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$$

constant v depends on the tension & linear density of string.

v is called "velocity" because it is the velocity w/ which a disturbance @ one end of the string travels along the string.

boundary conditions are that

$$y(0,t)=0 \quad , \quad y(L,t)=0$$

Now solve w/ separation of variables technique:

$$y(x,t) = X(x)\,T(t)$$

$$\Rightarrow \frac{\partial^2}{\partial x^2}\left[XT\right] = \frac{1}{v^2}\frac{\partial^2}{\partial t^2}\left[XT\right]$$

$$\Leftrightarrow \quad T\frac{\partial^2 X}{\partial x^2} = \frac{1}{v^2}\,X\,\frac{\partial^2 T}{\partial t^2}$$

$$\Leftrightarrow \quad \frac{\partial^2 X}{\partial x^2}\Big/X = \frac{1}{v^2}\,\frac{\partial^2 T}{\partial t^2}\Big/T$$

$\Rightarrow$ only solutions that can satisfy this for all $x$ & all $t$ is if

$$\frac{\partial^2 X}{\partial x^2} / X = \frac{1}{v^2} \frac{\partial^2 T}{\partial t^2} / T = \text{const.}$$

$$= -k^2$$

(separation constant)

$$\Rightarrow \frac{\partial^2 X}{\partial x^2} + k^2 X = 0 \quad \& \quad \frac{\partial^2 T}{\partial t^2} + k^2 v^2 T = 0$$

Take $k = \frac{2\pi}{\lambda} = \frac{2\pi \nu}{v} = \frac{\omega}{v}$ = wave number

$\nu$ ← frequency

$v$ ← velocity

$\omega$ ← angular freq.

$v = \lambda \nu$

$v$ ← velocity

$\lambda$ ← wavelength

$\nu$ ← frequency

so the solutions have the form

$$X(x) = A \sin kx + B \cos kx \qquad \left(\begin{array}{l}\text{osc.}\\ \text{space}\end{array}\right)$$

$$T(t) = C \sin \omega t + D \cos \omega t \qquad \left(\begin{array}{l}\text{osc,}\\ \text{time}\end{array}\right)$$

Now apply boundary conditions:

$$X(0) = 0 \Rightarrow B = 0$$

$$X(L) = 0 \Rightarrow k_n = \frac{n\pi}{L} \Rightarrow \omega_n = k_n v$$

$\Rightarrow$ solutions are

$$y_n^{(1)}(x,t) = b_n \sin(k_n x) \sin(\omega_n t)$$

or

$$y_n^{(2)}(x,t) = c_n \sin(k_n x) \cos(\omega_n t)$$

$\Rightarrow$ general solution is

$$y(x,t) = \sum_{n=0}^{\infty} b_n \sin(k_n x) \sin(\omega_n t) + c_n \sin(k_n x) \cos(\omega_n t).$$

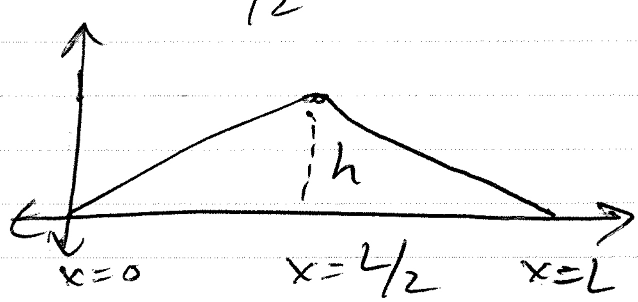
can apply some boundary conditions in time:

1) suppose string is plucked

& 2) initial velocity is zero

then initial configuration is

$$y(x,0) = f(x) = \begin{cases} hx/L/2 & \text{for } 0 < x < L/2 \\ \frac{h}{L/2}(L-x) & \text{for } \frac{L}{2} < x < L \end{cases}$$




Also, $\left.\frac{\partial y(x,t)}{\partial t}\right|_{t=0} = 0 \quad \forall x$

(initial velocity is zero)

$\Rightarrow b_n = 0 \quad \forall n$

b/c $\frac{\partial}{\partial t} \sin(\omega t)\Big|_{t=0} \propto \cos \omega t\Big|_{t=0} = 1$

$$\Rightarrow y(x,t) = \sum_{n=0}^{\infty} c_n \sin(k_n x) \cos(\omega_n t)$$

Also the other ~~time~~ boundary condition implies that

$$y(x,0) = f(x) = \sum_{n=0}^{\infty} c_n \sin(k_n x)$$

$$= \sum_{n=0}^{\infty} \underbrace{\left[\sqrt{\frac{L}{2}}\, c_n\right]}_{d_n} \left[\sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right)\right]$$

$$\Rightarrow d_n = \int_0^L f(x) \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right) dx$$

$$= \sqrt{\frac{2}{L}} \int_0^{L/2} \frac{h}{L/2}\, x \sin\left(\frac{n\pi x}{L}\right) dx$$

$$+ \sqrt{\frac{2}{L}} \int_{L/2}^{L} h/L/2\, (L-x) \sin\left(\frac{n\pi x}{L}\right) dx$$

$$= \sqrt{\frac{2}{L}} \cdot \frac{2h}{L} \left[ \int_0^{L/2} x \sin\left(\frac{n\pi x}{L}\right) dx + \int_{L/2}^{L} (L-x) \sin\left(\frac{n\pi x}{L}\right) dx \right]$$

change of variable $u = \frac{n\pi x}{L}$

$$\Rightarrow = \sqrt{\frac{2}{L}} \cdot \frac{2h}{L} \left(\frac{n\pi}{L}\right)^{-2} \left[ \int_0^{n\pi/2} u \sin u \, du + \int_{\frac{n\pi}{2}}^{n\pi} (n\pi - u) \sin u \, du \right]$$

$$= \sqrt{\frac{2}{L}}\ \frac{2hL}{n^2\pi^2} \left[ \sin\left(\frac{n\pi}{2}\right) - \frac{n\pi}{2} \cos\left(\frac{n\pi}{2}\right) + \sin\left(\frac{n\pi}{2}\right) + \frac{n\pi}{2} \cos\left(\frac{n\pi}{2}\right) - \sin(n\pi) \right]$$

$$= \sqrt{\frac{2}{L}} \frac{4hL}{n^2\pi^2} \sin\left(\frac{n\pi}{2}\right)$$

$$\Rightarrow c_n = \sqrt{\frac{2}{L}} d_n = \frac{2}{L} \cdot \frac{4hL}{n^2\pi^2} \sin\left(\frac{n\pi}{2}\right)$$

$$= \frac{8h}{n^2\pi^2} \sin\left(\frac{n\pi}{2}\right)$$

$$\Rightarrow \sin\left(\frac{n\pi}{2}\right) = 0 \quad \forall\ n \text{ even}$$

$$\sin\left(\frac{n\pi}{2}\right) = \pm 1 \quad \forall\ n \text{ odd}$$

$$n = 2m+1 \Rightarrow \sin\left(\frac{(2m+1)\pi}{2}\right) = (-1)^m$$

$$\Rightarrow c_{2m+1} = (-1)^m \frac{8h}{(2m+1)^2\pi^2}$$

$\Rightarrow$ solution is

$$y(x,t) = \sum_{m=0}^{\infty} c_{2m+1} \sin\left(\frac{(2m+1)\pi x}{L}\right)\cos\left(\frac{(2m+1)\pi v t}{L}\right)$$

$$= \frac{8h}{\pi^2} \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)^2} \sin\left(\frac{(2m+1)\pi x}{L}\right)\cos\left(\frac{(2m+1)\pi v t}{L}\right)$$

Show Mathematica.