Lecture 24

Partial differential equations

- Many problems involve the solution of PDEs. Also, same PDE may apply to a variety of physical problems.
- We list interesting ones here

1) Laplace's equation

$$\nabla^2 u = 0$$

u could be gravitational potential in a region containing no mass, electrostatic potential in a charge-free region, steady-state temperature in a region containing no heat sources.

2) Poisson's equation:

$$\nabla^2 u = f(x,y,z)$$

same as Laplace's, but there are sources of mass, charge, heat, etc. $f(x,y,z)$ is the source density.

3) Diffusion or heat flow equation:

$$\nabla^2 u = \frac{1}{\alpha^2} \frac{\partial u}{\partial t}$$

- u could be the non-steady state temperature in a region w/ no heat sources.
- α^2 is the diffusivity constant.

4) Wave equation:

$$\nabla^2 u = \frac{1}{v^2} \frac{\partial^2 u}{\partial t^2}$$

u could be

a) displacement from equilibrium of a vibrating string.

b) current or potential along a transmission line.

c) component of $\vec{E}$ or $\vec{B}$ in an electromagnetic wave.

v is the propagation speed.

5) Helmholtz equation:

$$\nabla^2 F + k^2 F = 0$$

- F represents spatial part of solution of diffusion or wave equation.

6) Schrödinger equation:

$$-\frac{\hbar^2}{2m}\nabla^2\psi + V\psi = i\hbar\frac{\partial}{\partial t}\psi$$

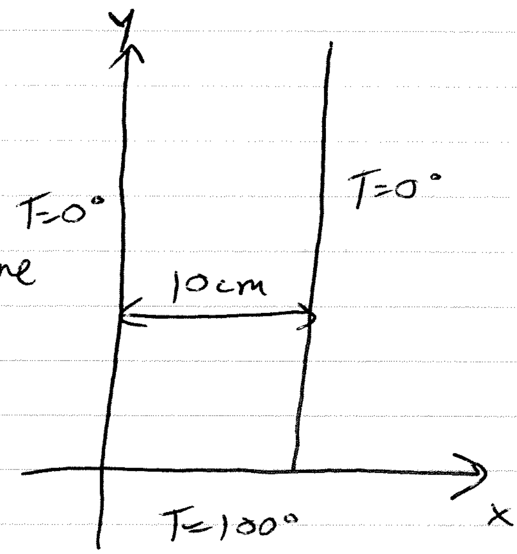
wave equation of quantum mechanics.

absolute square of Ψ is equal to the probability of the position of a particle.

- Principally concerned w/ solving these equations, not in deriving them.
- Could just accept them as being consistent w/ experimental observation.
- Main technique will be separation of variables.

Temperature in a rectangular plate.

Goal: Find the steady-state temperature distribution inside the plate.



(Mathematically equivalent to the problem of finding electrostatic potential if temperatures are replaced by potentials.)

Suppose @ first that plate is semi-infinite (extends to ∞ in y direction)

temperature $T(x,y)$ satisfies Laplace's equation inside the plate (no sources of heat)

$\nabla^2 T = 0$

Equivalently, $$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

use rectangular coordinates & ignore z b/c this is a 2D problem.

1st idea is to try separation of variables approach:

$$T(x,y) = X(x)\, Y(y)$$

Don't know whether this will be a solution, but it will be helpful for arriving at one.

– Substitute this guess into Laplace's equation to get

$$\frac{\partial^2}{\partial x^2}\left[X(x)\, Y(y)\right] + \frac{\partial^2}{\partial y^2}\left[X(x)\, Y(x)\right] = 0$$

$$\Leftrightarrow \quad Y \frac{\partial^2 X}{\partial x^2} + X \frac{\partial^2}{\partial y^2} Y = 0$$

Divide by $X \cdot Y$ to get

$$\frac{1}{X}\frac{\partial^2}{\partial x^2}X + \frac{1}{Y}\frac{\partial^2}{\partial y^2}Y = 0$$

$$\Leftrightarrow \frac{1}{X}\frac{\partial^2}{\partial x^2}X = -\frac{1}{Y}\frac{\partial^2}{\partial y^2}Y$$

Equation places strong constraints on possible solutions.

Consider: for fixed y, equation must hold & $\frac{1}{X}\frac{\partial^2}{\partial x^2}X$ should be equal to a constant, depending only on y.

However, $\frac{1}{X}\frac{\partial^2}{\partial x^2}X$ has no dependence on y & so it is equal to the same constant $\forall y$.

By the same reasoning, $\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2}$ is equal to a constant + due to the equation, they must be equal to the same constant.

$$\Rightarrow \quad \frac{1}{X}\frac{\partial^2}{\partial x^2}X = -\frac{1}{Y}\frac{\partial^2}{\partial y^2}Y = -k^2$$

w/ $k \geq 0$ $\quad k^2$ is called <u>separation constant</u>.

$\Rightarrow$ PDE splits into 2 ordinary diff. eq's

$$\frac{1}{X}\frac{\partial^2}{\partial x^2}X = -k^2 \quad \& \quad -\frac{1}{Y}\frac{\partial^2}{\partial y^2}Y = -k^2$$

$\Rightarrow$ $X'' + k^2 X = 0$ (simple harmonic oscillator)

$Y'' - k^2 Y = 0$ (exponential decay)

Solutions of the individual equations are

$$X(x) = A \sin kx + B \cos kx$$

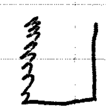
$$Y(y) = Ce^{ky} + De^{-ky}$$

There are two second-order diff. eq.'s

w/ two solutions each

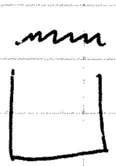
$\Rightarrow$ 4 constants of integration &

4 boundary conditions to solve

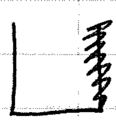
 I $\quad T(0,y) = X(0)\,Y(y) = 0 \Rightarrow X(0) = 0$

 II $\quad T(x,0) = X(x)\,Y(0) = 100°$

$\Rightarrow X(x)\,Y(0) = 100° = T_0$

 III $\quad T(x,\infty) = X(x)\,Y(\infty) = 0$

$\Rightarrow Y(\infty) = 0$

 IV $\quad T(L,y) = X(L)\,Y(y) = 0 \Rightarrow X(L) = 0$

Now apply them:

I $\quad X(0)=0 \quad \Rightarrow \quad B=0$ since

$\cos(0)=1\neq 0$

II ?

III $\quad Y(\infty)=0 \quad \Rightarrow \quad C=0$ since $e^{\infty}\neq 0$

IV $\quad X(L)=0 \quad \Rightarrow \quad A\sin kL=0$

I + IV imply $\quad X(x)=A\sin kx$

where $kL=n\pi$

III $\Rightarrow \quad Y(y) = De^{-kx}$ $\quad \Rightarrow \quad k_n=\frac{n\pi}{L}$

General solution is then

$$T(x,y) = \underbrace{AD}_{b_n}\, e^{-k_n y}\sin(k_n x)$$

Thus, the temperature drops off exponentially in y but oscillates in x.

Back to the boundary condition II

$$T(x,0) = 100° = b_n e^{-k_n y} \sin(k_n x)\Big|_{y=0}$$

$$\Rightarrow \quad 100° = b_n \sin(k_n x)$$

This cannot be true for any single b_n since $\sin(k_n x) \neq$ constant

- But the functions $\{\sin k_n x\}_n$ are a complete, orthogonal set that span the solution space. So the most general solution has the form

$$T(x,y) = \sum_n \underbrace{b_n e^{-k_n y}}_{c_n} \sin(k_n x)$$

which is a superposition of individual solutions by linearity.

- Then treat as a Fourier sine series in x:

$$T(x) = \sum_n c_n \sin(k_n x)$$

So the boundary condition II becomes

$$T(x, 0) = \sum_{n=0}^{\infty} b_n e^0 \sin(k_n x) = T_0$$

rewrite slightly as follows to use orthonormality:

$$\sum_{n=0}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

$$= \sum_{n=0}^{\infty} \underbrace{\left(\sqrt{\frac{L}{2}}\, b_n\right)}_{c_n} \cdot \left(\sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right)\right)$$

then

$$\int_0^L \sqrt{\frac{2}{L}} \sin\left(\frac{m\pi x}{L}\right) \cdot \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right) dx = \delta_{n,m}$$

$$\Rightarrow c_n = \int_0^L \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right) T_0 \, dx$$

$$= T_0 \sqrt{\frac{2}{L}} \int_0^L \sin\left(\frac{n\pi x}{L}\right) dx$$

$$= T_0 \sqrt{\frac{2}{L}} \left[\frac{-L}{n\pi} \cos\left(\frac{n\pi x}{L}\right)\right]\Bigg|_0^L$$

$$= T_0 \sqrt{\frac{2}{L}} \frac{L}{n\pi} \left[1 - \cos n\pi\right]$$

$$= T_0 \frac{\sqrt{2L}}{n\pi} \begin{cases} 0 & : \text{even } n \\ 2 & : \text{odd } n \end{cases}$$

$$\Rightarrow b_n = \sqrt{\frac{2}{L}} c_n = 2T_0 \sqrt{\frac{2}{L}} \cdot \sqrt{2L} \frac{1}{n\pi} \cdot \begin{cases} 0 & : \text{even } n \\ 1 & : \text{odd } n \end{cases}$$

$$= \frac{4T_0}{n\pi} \begin{cases} 0 & : \text{even } n \\ 1 & : \text{odd } n \end{cases}$$

only odd terms survive.

$\Rightarrow$ final solution is

$$T(x,y) = \frac{400}{\pi} \sum_{m=0}^{\infty} \frac{1}{2m+1} \exp\left(-\frac{(2m+1)\pi y}{L}\right) \cdot \sin\left((2m+1)\pi x/L\right)$$

evaluate w/ Mathematica