Lecture 22

orthogonality of Bessel functions

Consider $\sin x$ & $\cos x$.

these are similar to $J_p(x)$ & $Y_p(x)$.

$\sin x = 0 \iff x = n\pi$ for $n \in \mathbb{Z}$

same as

$\sin(n\pi x) \iff x \in \mathbb{Z}$

$$\sin''(n\pi x) = -(n\pi)^2 \sin(n\pi x)$$

$$\Rightarrow \quad \sin''(n\pi x) + (n\pi)^2 \sin(n\pi x) = 0$$

Now consider $J_p(x)$ & $Y_p(x)$

As stated above, these are like

$\sin x$ & $\cos x$, respectively

the roots of $J_p(x)$ are not integers but we can enumerate them as $J_p(\alpha_p)=0$, $J_p(\beta_p)=0$, ...

where $\alpha_p, \beta_p, \gamma_p$ are the roots of $J_p(x)$

$\Rightarrow$ @ $x=1$ $\quad J_p(\alpha_p x)=0$,

$$J_p(\beta_p x)=0, \ldots$$

Recall that $\{\sin(n\pi x)\}_n$ form an orthnormal set over $[0,1]$.

that is, $\int_0^1 \sin(n\pi x)\sin(m\pi x)\,dx = 0$

if $n \neq m$

& $\int_0^1 \sin(n\pi x)\sin(m\pi x)\,dx = 1/2$

if $n=m$

Our goal is to prove that

$$\int_0^1 x\, J_p(\alpha_p x)\, J_p(\beta_p x)\, dx = 0$$

for $\alpha_p \neq \beta_p$

Consider that

$$x^2 J_p''(x) + x J_p'(x) + (x^2 - p^2) J_p(x) = 0$$

~~can rescale by roots α_p or β_p as $\frac{d}{dx} J_p(x)$~~

Now substitute $z = \alpha x$ to get

$$z^2 \frac{d^2}{dz^2} J_p(z) + z \frac{d}{dz} J_p(z) + (z^2 - p^2) J_p(z) = 0$$

use chain rule to get

$$\frac{d J_p(z)}{dz} = \frac{dx}{dz} \frac{d}{dx} J_p(\alpha x) = \frac{1}{\alpha} \frac{d}{dx} J_p(\alpha x)$$

$$(\alpha \cancel{x})^2 \frac{1}{\alpha^2}\frac{d^2}{dx^2} J_p(\alpha x) + \alpha x \frac{1}{\alpha}\frac{d}{dx} J_p(\alpha x) + (\alpha^2 x^2 - p^2) J_p(\alpha x) = 0$$

$$\Rightarrow \quad x^2 \frac{d^2}{dx^2} J_p(\alpha x) + x \frac{d}{dx} J_p(\alpha x) + (\alpha^2 x^2 - p^2) J_p(\alpha x) = 0$$

Now use $x \cdot \frac{d}{dx}\left[x \frac{d}{dx} y\right] = x \frac{dy}{dx} + x^2 \frac{d^2 y}{dx^2}$

to rewrite as

$$x \frac{d}{dx}\left[x \cdot \frac{d}{dx} J_p(\alpha x)\right] + (\alpha^2 x^2 - p^2) J_p(\alpha x) = 0$$

same for β

$$x \frac{d}{dx}\left[x \frac{d}{dx} J_p(\beta x)\right] + (\beta^2 x^2 - p^2) J_p(\beta x) = 0$$

Recall that α & β are arbitrary roots of $J_p(x)$.

Use $u(x) \equiv J_p(\alpha x)$

$v(x) \equiv J_p(\beta x)$

to simplify notation.

Goal is to prove that

$$\int_0^1 x\, J_p(\alpha x)\, J_p(\beta x)\, dx = \frac{1}{2}\left[J_p'(\alpha)\right]^2 \cdot \delta_{\alpha,\beta}$$

where $\delta_{\alpha,\beta} = \begin{cases} 0 & \text{if } \alpha \neq \beta \\ 1 & \text{if } \alpha = \beta \end{cases}$

Proof: Suppose $\alpha \neq \beta$.

we have that

$$x\left[x u'\right]' + \left(\alpha^2 x^2 - p^2\right) u = 0$$

$$x\left[x v'\right]' + \left(\beta^2 x^2 - p^2\right) v = 0$$

Multiply 1st by v, 2nd by u subtract & divide by x to get

6

$$v(xu')' - u(xv')' + (\alpha^2-\beta^2)\,xuv = 0$$

can rewrite as

$$\frac{d}{dx}\left[vxu' - uxv'\right] + (\alpha^2-\beta^2)\,xuv = 0$$

Now integrate from 0 ~~to~~ 1 to get

$$\underbrace{\left(vxu' - uxv'\right)\Big|_0^1}_{} + (\alpha^2-\beta^2)\int_0^1 xuv\,dx = 0$$

lower limit = 0 b/c x is present & v, u', u, v' are finite

upper limit: recall that

$u = J_p(\alpha) = 0$ & $v = J_p(\beta) = 0$

since α & β are zeros of $J_p(x)$.

Then we get

$$(\alpha^2-\beta^2)\int_0^1 x\,u\,v\,dx=0$$

or

$$(\alpha^2-\beta^2)\int_0^1 x\,J_p(\alpha x)\,J_p(\beta x)\,dx=0$$

$$\Rightarrow \int_0^1 x\,J_p(\alpha x)\,J_p(\beta x)\,dx=0$$

if $\alpha\neq\beta$.

To see the result for $\alpha=\beta$, we employ a limiting arguement. Let α be a root of $J_p(x)$ & let $\beta\approx\alpha$ w/ β not a root.

- Note that

$$(vxu'-uxv')\Big|_0^1+(\alpha^2-\beta^2)\int_0^1 x\,u\,v\,dx=0$$

still holds but $J_p(\beta)\neq 0$ & $J_p(\alpha)=0$.

rewrite as

$$\left[J_p(\beta x) \times J_p'(\alpha x) - J_p(\alpha x) \times J_p'(\beta x)\right]\Big|_0^1$$

$$+ (\alpha^2 - \beta^2) \int_0^1 x \; J_p(\alpha x)\, J_p(\beta x)\, dx = 0$$

lower limit of 1st term $= 0$

upper limit is $= J_p(\beta) J_p'(\alpha) \cdot \alpha$

b/c $J_p(\alpha \cdot 1) = 0$, so right term goes away

$$\phi \quad \lim_{x \to 1} \frac{d}{dx} J_p(\alpha x)$$

Set $z = \alpha x$

$$= \lim_{x \to 1} \frac{dz}{dx} \cdot \frac{d}{dz} J_p(z)$$

$$= \alpha \cdot \lim_{x \to 1} \frac{d}{dz} J_p(z)$$

$$= \alpha \cdot J_p'(\alpha)$$

$$\Rightarrow \quad \frac{J_p(\beta)\cdot\alpha\cdot J_p'(\alpha)}{\beta^2-\alpha^2} = \int_0^1 x\, J_p(\alpha x)\cdot J_p(\beta x)\,dx$$

Now take $\lim_{\beta\to\alpha}$. Both top + bottom of LHS $\to 0$. Apply L'Hospital's rule & differentiate top + bottom wrt β:

$$\lim_{\beta\to\alpha} \frac{J_p'(\beta)\cdot\alpha\cdot J_p'(\alpha)}{2\beta} = \frac{1}{2}\left[J_p'(\alpha)\right]^2.$$

So we have proven that

$$\int_0^1 x\, J_p(\alpha x)\, J_p(\beta x)\,dx = \frac{1}{2}\left[J_p'(\alpha)\right]^2 \delta_{\alpha,\beta}$$

where α + β are roots of $J_p(x)$.

2 ways to look @ this

1) The functions

$$j_p(\alpha x) \equiv \sqrt{2}\,[J_p'(\alpha)]^{-1} \cdot \sqrt{x}\, J_p(\alpha x)$$

where α is a root

are othonormal on $x \in [0,1]$.

That is

$$\int_0^1 j_p(\alpha_p x)\, j_p(\beta_p x)\, dx = \delta_{\alpha,\beta}$$

As such, we can expand well-behaved functions in terms of Bessel's functions:

$$f(x) = \sum_\alpha c_\alpha\, j_p(\alpha x)$$

where

$$c_\alpha = \int_0^1 j_p(\alpha x)\, f(x)\, dx.$$

The functions

$$q_p(\alpha_p|x) \equiv \sqrt{2}\,[J_p'(\alpha)]^{-1} J_p(\alpha x)$$

are orthonormal w/ respect to weight function $\rho(x) = x$, s.t.,

$$\int_0^1 \rho(x)\, q_p(\alpha_p|x)\, q_p(\beta_p x)\, dx = \delta_{\alpha,\beta}$$

Now show page 604
Table of Asymptotic forms

Fuchs's theorem:

Now we've discussed two examples of differential equations, Legendre & Bessel, solvable by the Frobenius method of generalized power series.

- There are other examples of these kinds of functions:
  - they are the solutions of parametrized diff. eq's.
  - have orthogonality properties & other functions can be expanded in terms of them.
  - generating functions for them.
  - physical problems whose solutions involve them.
- Might wonder if all diff. eq's might be solved in this way.
- General theorem due to Fuchs states when this is possible:

Fuchs's theorem for 2nd order diff-eq's is

$$y'' + f(x)\, y' + g(x)\, y = 0$$

If $x f(x)$ & $x^2 g(x)$ are expandable in convergent power series, $\sum_{n=0}^{\infty} a_n x^n$

then diff. eq. is regular at the origin.

Theorem of Fuchs says that these conditions are necessary & sufficient for solution to consist of

1) two Frobenius series, i.e.,

$$y(x) = x^s \sum_{n=0}^{\infty} c_n x^n + x^t \sum_{n=0}^{\infty} d_n x^n$$

or

2) one solution $S_1(x)$ being a Frobenius series & a second solution $S_1(x) \ln x + S_2(x)$ w/ $S_2(x)$ Frobenius.