1

Lecture 21

Spherical Bessel functions

- If $p = \frac{(2n+1)}{2} = n + 1/2$ w/ n an integer

then $J_p(x)$ & $N_p(x)$ are called Bessel functions of half-odd integer order & can be written in terms of $\sin x$, $\cos x$, & x^k.

- spherical Bessel functions are

$j_n(x)$, $y_n(x)$, $h_n^{(1)}(x)$, $h_n^{(2)}(x)$

for $n = 0, 1, 2, \ldots$ & are given by

$$j_n(x) = \sqrt{\frac{\pi}{2x}}\, J_{\frac{2n+1}{2}}(x)$$

$$= x^n \left(-\frac{1}{x}\frac{d}{dx}\right)^n \left(\frac{\sin x}{x}\right)$$

$$y_n(x) = \sqrt{\frac{\pi}{2x}}\, Y_{\frac{2n+1}{2}}(x)$$

$$= -x^n \left(-\frac{1}{x}\frac{d}{dx}\right)^n \left(\frac{\cos x}{x}\right)$$

$$h^{(1)}(x) = j_n(x) + i\, y_n(x)$$

$$h^{(2)}(x) = j_n(x) - i\, y_n(x)$$

they arise in vibrational problems when spherical coordinates are used.

(example: finding steady-state temperature inside a sphere w/ surface temperature different from internal temperature).

Kelvin functions

show up in heat flow &
wave equation w/ dissipation or loss

$$x^2 y'' + xy' - ix^2 y = 0$$

loss term

Kelvin's diff. eq.

can compare w/ general Bessel equation to get

$$x^2 y'' + (1-2a) xy' + \left[(bcx^c)^2 + (a^2 - p^2c^2) \right] y = 0$$

to get

$$1 - 2a = 1 \Rightarrow a = 0$$

$$a^2 - p^2c^2 = 0 \Rightarrow p^2c^2 = 0 \quad \begin{matrix} 2c = 2 \\ \Rightarrow c = 1 \text{ \& } p = 0 \end{matrix}$$

$$b^2 = -i \Rightarrow b = \sqrt{-1}\,\sqrt{i} = i^{3/2}$$

$$\Rightarrow y(x) = Z_0(i^{3/2}x)$$

$$Z_0 \in \{J_0, Y_0\}$$

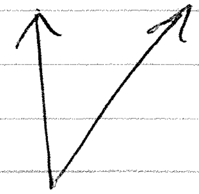
this is complex

It is customary to separate into real & imaginary parts:

$$J_0(i^{3/2}x) = \text{ber}\,x + i\,\text{bei}\,x$$

Airy's diff. eq. arises in E&M, optics, QM.

e.g. solution to Schrödinger's equation for a particle confined w/in a triangular potential well



5

$$y'' - xy = 0$$

write as

$$x^2y'' + 0\cdot y' + \left[-x^3 + 0\right]y = 0$$

& compare to

$$x^2y'' + (1-2a)\,xy' + \left[(bcx^c)^2 + a^2 - p^2c^2\right]y = 0$$

$$1 - 2a = 0 \Rightarrow a = 1/2$$

$$2c = 3 \Rightarrow c = 3/2$$

$$(1/2)^2 - p^2\,(3/2)^2 = 0$$

$$\Rightarrow \quad p^2 = \frac{(1/2)^2}{(3/2)^2}$$

$$\Rightarrow \quad p = \frac{1/2}{3/2} = 1/3$$

$$(bc)^2 = -1$$

$$\Rightarrow bc = i \Rightarrow \quad b = 2/3\, i.$$

+ solutions are

$$x^a Z_p[b x^c] = \sqrt{x}\; Z_{1/3}[2/3\, i x^{3/2}]$$

where $Z_{1/3} \in \{J_{1/3}, Y_{1/3}, ..\}$

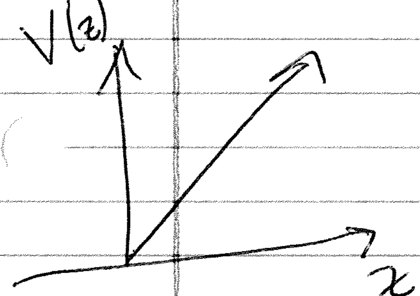
A particular linear combination gives

$$Ai[x] = \frac{1}{\pi}\sqrt{\frac{x}{3}}\; K_{1/3}\left(\tfrac{2}{3}x^{3/2}\right)$$

$$Bi[x] = \sqrt{\frac{x}{3}}\left[I_{-1/3}\left(\tfrac{2}{3}x^{3/2}\right) + I_{1/3}\left(\tfrac{2}{3}x^{3/2}\right)\right]$$

quantum bouncing ball

- quantum particle bouncing on a perfectly reflecting surface under the influence of gravity.
- particle is subjected to the potential



$$V(z) = \begin{cases} mgz & : z \geq 0 \\ \infty & : z < 0 \end{cases}$$

time independent Schrödinger equation is given by

$$-\frac{\hbar^2}{2m} \psi''(z) + V(z)\,\psi(z) = E\,\psi(z)$$

where z is the position of the particle & $V(z)$ is the potential energy.

(goal is to find stationary energy eigenstates, which can be used to figure out dynamics) the interpretation of the wave function is that, upon squaring it, you have the probability that the particle is found in a particular location.

- Apply the boundary conditions

$$\psi(0) = 0 \quad \& \quad \psi(\infty) = 0$$

Interpretation: zero probability to find quantum particle @ these locations.

Then rewrite Schrödinger equation as

$$\psi''(z) + \left[\frac{2mE}{\hbar^2} - \frac{2m^2 g}{\hbar^2} z\right] \psi(z) = 0$$

rewrite again as

$$\psi''(z) + \left[\varepsilon - \gamma z\right] \psi(z) = 0$$

where $\varepsilon = \frac{2mE}{\hbar^2}$ is scaled

energy w/ units $\frac{1}{[\text{length}]^2}$

& $\gamma \equiv \frac{2m^2 g}{\hbar^2}$ has units $\frac{1}{[\text{length}]^3}$

Make the substitution

$$x = \frac{z\gamma - \varepsilon}{\gamma^{2/3}} \quad \text{(dimensionless)}$$

& note that $\frac{d}{dz} = \frac{dx}{dz}\frac{d}{dx}$

$$= \gamma^{1/3}\frac{d}{dx}$$

$$\Rightarrow \frac{d^2}{dz^2} = \gamma^{2/3}\frac{d^2}{dx^2}$$

& $\varepsilon - \gamma z = -\gamma^{2/3} x$

to get that $\gamma^{2/3}\psi''(x) - \gamma^{2/3} x\psi(x) = 0$

$$\Rightarrow \quad \psi''(x) - x\psi(x) = 0$$

which is the Airy diff. eq.

General solution is

$$\psi(x) = A\,Ai(x) + B \cdot Bi(x)$$

where Ai & Bi are the Airy Bessel functions.

can also write as

$$\psi(z) = A \cdot Ai\left[\frac{z\gamma - \varepsilon}{\gamma^{2/3}}\right] + B \cdot Bi\left[\frac{z\gamma - \varepsilon}{\gamma^{2/3}}\right]$$

To simplify analysis, choose mass units such that $\gamma = 1$

$$\text{\&}\ \psi(z) = A \cdot Ai\left[z - \varepsilon\right] + B \cdot Bi\left[z - \varepsilon\right]$$

Note also that $V(z) = z$ in these units.

Can show that for fixed ε,

$$\lim_{\cancel{z} \to \infty} A_i[\cancel{z}] = 0$$

but $\lim_{z \to \infty} B_i[z] = \infty$

& so B_i cannot be a solution

Boundary condition gives $B=0$ &

So solution is $\psi(z) = A \cdot A_i[z-\varepsilon]$

Applying other boundary condition $\psi(0) = 0$ gives that

$$A \cdot A_i[-\varepsilon] = 0$$

Since we require $A \neq 0$ to have a nontrivial solution, this means that

$$Ai[-\varepsilon] = 0$$

that is, the only allowed energies of the quantum particle are such that $-\varepsilon$ is a root of Ai.

$$\varepsilon \in \{\varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots\}$$

very different from a classical particle which can take on a continuum of energies.

- This is thus a quantization condition for energies.
- can then numerically solve for energies as (1st three of them)

$$\varepsilon_1 \approx 2.34$$
$$\varepsilon_2 \approx 4.09$$
$$\varepsilon_3 \approx 5.52$$

energy eigenfunctions are then

$$\psi_i(z) = A_i \cdot Ai[z - \varepsilon_i] \text{ for } i = 1, 2, 3$$

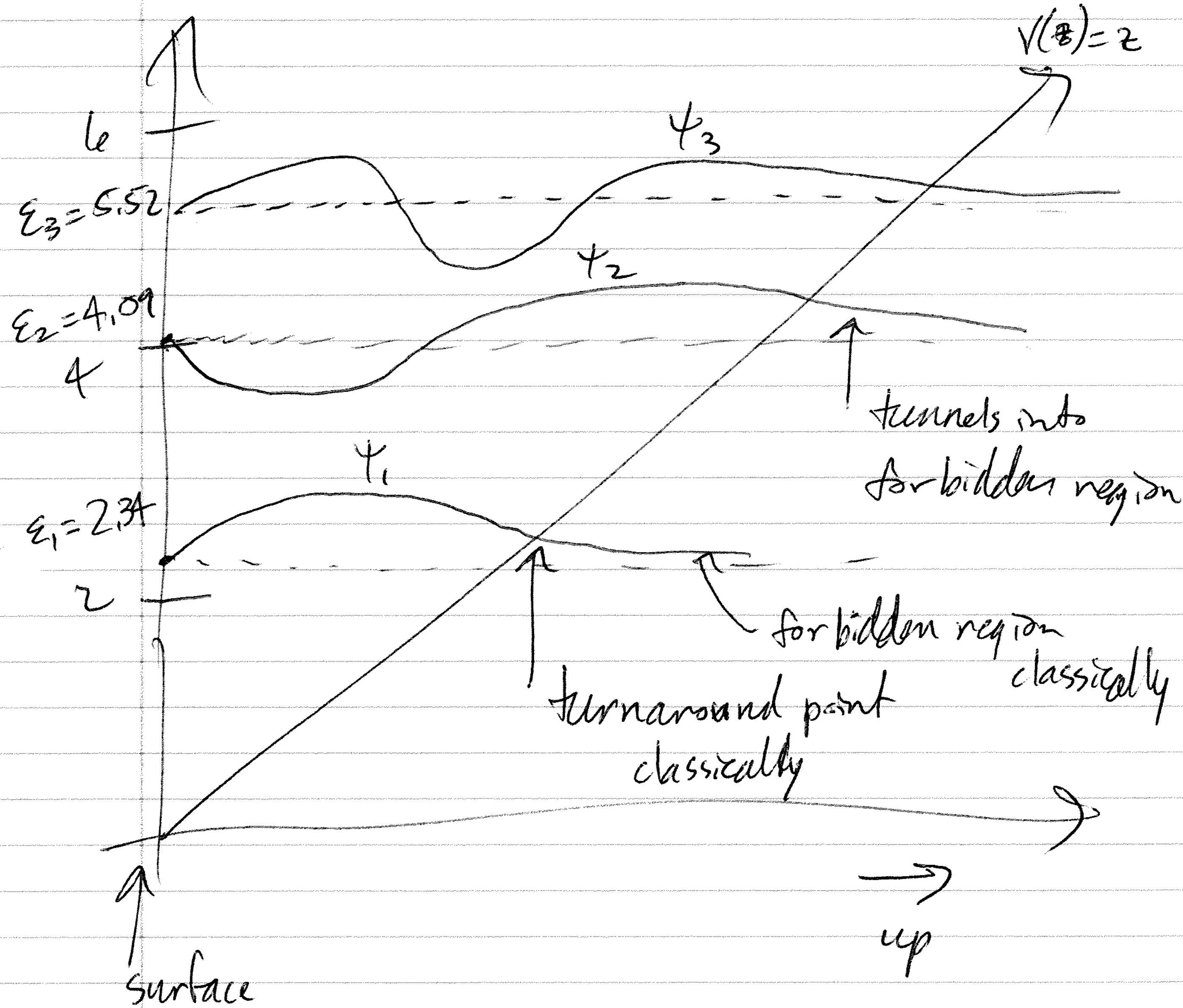
to find normalization constants A_i apply the Born rule

$$\int_0^\infty |\psi_i(z)|^2 = 1$$

From here, it is typical to plot

$$\Psi_i[z] = \varepsilon_i + \psi_i(z)$$

so that the horizontal axis for each eigenfunction lies at its own energy level. Plotting w/ $V(z) = z$, we get



Classically, the line $V(z) = z$ corresponds to the point where $E = mgh$ & all energy is potential energy at the highest point, where the ball turns around quantum ball can be found above the classical turnaround - it "tunnels" a bit into classically forbidden region.

Also, the classical ball can have any energy, whereas the quantum ball has only discrete energies.