Lecture 20

1

Recall Bessel's diff. eq.:

$$x^2 y'' + x y' + (x^2 - p^2) y = 0$$

Last time we found two solutions using generalized power series method:

$$J_p(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(n+p+1)} \left(\frac{x}{2}\right)^{2n+p}$$

for $p \geq 0$

&

$$J_{-p}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(n-p+1)} \left(\frac{x}{2}\right)^{2n-p}$$

w/ general solution

$$A\, J_p(x) + B\, J_{-p}(x)$$

for constants A & B.

(2)

If p is an integer, then the first few terms of J_{-p} are equal to zero because $\Gamma(n-p+1)$ in the denominator is Γ of a negative integer, which is $= \infty$.

- Can then show that

$$J_{-p}(x) = (-1)^p J_p(x) \text{ for integer } p.$$

Thus $J_{-p}(x)$ is not linearly independent for integer p.

Let us prove this:

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$$J_{-m} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(n+1-m)} \left(\frac{x}{2}\right)^{2n-m}$$

$$\Gamma(0) = \Gamma(-1) = \Gamma(-2) = \dots = \infty$$

$\Rightarrow$ reciprocals are equal to zero.

$\Rightarrow$ All terms in series starting from $n+1-m \leq 0$ & so series starts @ $n+1-m=1$

$$\Rightarrow n \in \{m, m+1, m+2, \dots\}$$

$$\Rightarrow J_{-m} = \sum_{n=m}^{\infty} \frac{(-1)^n}{n!\,(n-m)!} \left(\frac{x}{2}\right)^{2n-m}$$

let $l = n-m$

$\Rightarrow n = m+l$

$$= \sum_{l=0}^{\infty} \frac{(-1)^{l+m}}{(m+l)!\,l!} \left(\frac{x}{2}\right)^{2(m+l)-m}$$

$$= (-1)^m \sum_{l=0}^{\infty} \frac{(-1)^l}{(m+l)!\,l!} \left(\frac{x}{2}\right)^{2l+m}$$

$$= (-1)^m J_m(x)$$

- Even though $J_{-p}(x)$ is a satisfactory solution when p is non-integer, it is customary to use a linear combination of $J_p(x)$ & $J_{-p}(x)$ as the 2^{nd} solution.
- Similar to using $\sin x$ & $2\sin x - 3\cos x$ as solutions of $y'' + y = 0$ instead of $\sin x$ & $\cos x$.
- Combination which is used by convention is

$$N_p(x) = Y_p(x) = \frac{\cos(\pi p) J_p(x) - J_{-p}(x)}{\sin(\pi p)}$$

For integer p, this expression has an indeterminate form, so use L'Hospital & define

$$Y_m = N_m = \lim_{p \to m} \frac{\cos(\pi p) J_p - J_{-p}}{\sin(\pi p)}$$

Then find that

$$Y_m(x) = \frac{2}{\pi}\left[\ln(x/2) + \gamma\right] J_m(x) - \frac{1}{\pi}\sum_{n=0}^{m-1} \frac{(m-n-1)!}{n!}\left(\frac{x}{2}\right)^{2n-m}$$

where γ is Euler's constant:

$$\gamma \equiv \lim_{n\to\infty}\left[\sum_{m=1}^{n}\frac{1}{m} - \ln n\right]$$

$$\simeq 0.577$$

Graphs & zeros of Bessel functions

bring up Mathematica

roots do not occur @ regular intervals & need to be solved numerically.

Recursion Relations for Bessel functions

Several of them, one of which is

$$\frac{d}{dx}\left[x^p J_p(x)\right] = x^p J_{p-1}(x)$$

Proof for integer p:

$$x^p J_p(x) = x^p \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)\,\Gamma(n+1+p)} \left(\frac{x}{2}\right)^{2n+p}$$

$$= x^m \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\; n+m!} \left(\frac{x}{2}\right)^{2n+m}$$

$$= \left(2 \cdot \frac{x}{2}\right)^m \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\; n+m!} \left(\frac{x}{2}\right)^{2n+m}$$

$$= 2^m \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\; n+m!} \left(\frac{x}{2}\right)^{2(n+m)}$$

$$\Rightarrow \frac{d}{dx}\left[x^m J_m(x)\right]$$

$$= 2^m \sum_{n=0}^{\infty} \frac{(-1)^n \, 2(n+m)}{n!\,(n+m)!} \left(\frac{x}{2}\right)^{2(n+m)-1} \cdot \frac{1}{2}$$

$$= 2^m \left(\frac{x}{2}\right)^m \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,(n+m-1)!} \left(\frac{x}{2}\right)^{2n+m-1}$$

$$= x^m \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,(n+m-1)!} \left(\frac{x}{2}\right)^{2n+m-1}$$

$$= x^m J_{m-1}(x)$$

<u>Generalized Bessel diff. eq.:</u>

$$x^2 y'' + x(1-2a)y' + \left[(bc\,x^{c-1})^2 + \frac{a^2 - p^2c^2}{x^2}\right] y = 0$$

has solution

$$y(x) = x^a Z_p(bx^c) \quad \text{where } Z_p = \begin{cases} Y_p \\ J_p \end{cases}$$

Proof: let $y(x) = x^a u(z)$

where $z = bx^c$

$$\Rightarrow \frac{du}{dx} = \frac{du}{dz}\frac{dz}{dx} = u'(z)\, bc\, x^{c-1}$$

$$\Rightarrow y' = \frac{d}{dx}\left[x^a u\right] = a x^{a-1} u + x^a\, bc\, x^{c-1} u'(z)$$

$$\Rightarrow xy' = a x^a u(z) + bc\, x^{a+c} u'(z)$$

$$\Rightarrow y'' = \frac{d}{dx} y'(x) = a\cdot(a-1) x^{a-2} u + a\, x^{a-1} \frac{d}{dx}\left[u(z)\right] + bc\,(a+c-1)\, x^{a+c-2} \cdot u' + x^{a+c-1}\, bc\, \frac{d}{dx}\left[u'(z)\right]$$

$$= a(a-1) \cdot x^{a-2} u + a\, x^{a-1}\, bc\, x^{c-1} u'(z) + bc\,(a+c-1)\, x^{a+c-2} \cdot u'(z) + (bc)^2 x^{a+2(c-1)} \cdot u''(z)$$

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$$\Rightarrow x^2 y'' = a(a-1)\, x^a u(z) +$$
$$abc\, x^{a+c} u'(z) +$$
$$bc(a+c-1)\, x^{a+c} u'(z) +$$
$$(bc)^2\, x^{a+2c} u''(z)$$

Plug into generalized Bessel equation
& collect terms to get

$$(bc)^2 x^{a+2c} u''(z) + [abc + bc(a+c-1)] x^{a+c} u'(z)$$
$$+ a(a-1) x^a u(z) + (1-2a)[a x^a u(z) + bc x^{a+c} u']$$
$$+ [(bc\, x^c)^2 + (a^2 - p^2 c^2)] x^a u$$

After some algebra, can be reduced to

$$= c^2 x^a [z^2 u''(z) + z u'(z) + (z^2 - p^2) u(z)]$$

part in [·] is regular Bessel diff. eq.

$\Rightarrow$ whole expression $= 0$ if

$$u(z) = J_p(z) \text{ or } Y_p(z) = Z_p(z)$$

$$\Rightarrow y = x^a Z_p(z)$$
$$= x^a Z_p(bx^c)$$

<u>Example:</u> solve Airy's diff. eq.

$$y'' + 9xy = 0$$

$$\Rightarrow \quad x^2 y'' + 9x^3 y = 0$$

compare to

$$x^2 y'' + x(1-2a)\,y' +$$
$$\left[b^2 c^2 x^{2c} + \left(a^2 - p^2 c^2\right)\right] y = 0$$

Set $a = 1/2$

$c = 3/2$

$b^2 c^2 = 9 \Rightarrow b = 2$

$\Rightarrow$ then require $a^2 - p^2 c^2 = 0 \Rightarrow p = 1/3$

can read off solution as

$$y(x) = \sqrt{x}\; Z_{1/3}\left(2x^{3/2}\right)$$

$\Rightarrow$ general solution of

$$y(x) = A\sqrt{x}\, J_{1/3}\left[2x^{3/2}\right] + B\sqrt{x}\, Y_{1/3}\left[2x^{3/2}\right]$$

Other Bessel functions

- Recall that $y'' + k^2 y = 0$ has two real independent solutions $\cos kx$ & $\sin kx$.
- We can construct complex solutions via

$$e^{\pm ikx} = \cos kx \pm i \sin kx$$

- Similarly, Bessel's diff. eq. is

$$x^2 y'' + x y' + \left[x^2 - p^2\right] y = 0$$

has 2 real independent solutions $J_p(x)$ & $Y_p(x)$ which for large x look like

$$J_p(x) \underset{|x|>>1}{\approx} \sqrt{\frac{2}{\pi x}} \cos\left[x - \frac{p\pi}{2} - \frac{\pi}{4}\right]$$

$$Y_p(x) \underset{|x|>>1}{\approx} \sqrt{\frac{2}{\pi x}} \sin\left[x - \frac{p\pi}{2} - \frac{\pi}{4}\right]$$

we can construct complex solutions as

$$H_p^{\pm}(x) = J_p(x) \pm i\, Y_p(x)$$

called Hankel functions or Bessel functions of the 3rd kind.

Then for large x, we have

$$H_p^{(\pm)}(x) \underset{|x|>>1}{\approx} \sqrt{\frac{2}{\pi x}} \exp\left[\pm i\left(x - \frac{p\pi}{2} - \frac{\pi}{4}\right)\right]$$

Hyperbolic Bessel functions

$$y'' + k^2 y = 0 \qquad k \in \mathbb{R}$$

has solutions $\qquad e^{\pm ikx}$

If we instead let $k = iK$

then diff. eq. is

$$y'' - K^2 y = 0$$

which has solutions $e^{\pm Kx}$

or also $\quad \cosh(x) = \dfrac{e^{Kx} + e^{-Kx}}{2}$

$$\sinh(x) = \frac{e^{Kx} - e^{-Kx}}{2}$$

hyperbolic functions.

Note that

$$\sin(ix) = \frac{e^{i[ix]} - e^{-i[ix]}}{2i} = \frac{e^{-x} - e^{x}}{2i}$$

$$= i\left[\frac{e^{x} - e^{-x}}{2}\right] = i\sinh(x)$$

Similarly, if $x \in \mathbb{R}$

then $I_p(x) = i^{-p} J_p(ix)$

$$K_p(x) = \frac{\pi}{2} i^{p+1} H_p^{(+)}(ix)$$

are solutions to

$$x^2 y'' + xy' - (x^2+p^2)y = 0.$$