Lecture 18

- We previously looked at series of the form $y(x) = \sum_{n=0}^{\infty} a_n x^n$ for well-behaved functions @ the origin.

- But in physics, some functions are not well behaved, such as

$$V(r) = \frac{e^2}{r}$$

has a simple pole singularity @ $r=0$.

- We would like to handle such functions as solutions to diff. eq.'s.

- For example,

$$y(x) = \frac{\cos(x)}{x^2} = \frac{1}{x^2}\left[1 - \frac{1}{2}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \ldots\right]$$

$$= \underbrace{\frac{1}{x^2}}_{\text{singular}} - \overbrace{\frac{1}{2} + \frac{x^2}{4!} - \frac{x^4}{6!} + \ldots}^{\text{well behaved}}$$

or $y = \sqrt{x} \sin x$

$$= \sqrt{x} \left[x - \frac{1}{3!} x^3 + \frac{1}{5!} x^5 + \dots \right]$$

has no derivatives @ $x = 0$.

Both cases are covered by a series in the Frobenius form:

$$y(x) = x^s \sum_{n=0}^{\infty} a_n x^n \quad \text{where } s \in \mathbb{R}$$

For the examples, $s = -2$ & $s = 1/2$

As an example, consider solving the following diff. eq.:

$$x^2 y'' + 4xy' + (x^2+2)y = 0$$

For $|x| \ll 1$, we can neglect $x^2 \lll 1$

to get

$$4xy' + 2y \approx 0$$

$$\Rightarrow 2xy' \approx -y$$

$$\Rightarrow 2x\frac{dy}{dx} \approx -y \Rightarrow \frac{dy}{y} \approx -\frac{dx}{2x}$$

$$\Rightarrow \ln y \approx -\frac{1}{2}\ln x + c$$

$$\Rightarrow e^{\ln y} \approx e^{\ln x^{-1/2} + c}$$

$$\Rightarrow y = x^{-1/2} \cdot e^{c}$$

$$\Rightarrow y = A \cdot x^{-1/2} \quad \text{for } |x| \ll 1$$

Conclude that $y(0)$ & $y'(0)$ are not defined @ $x=0$ & so we should use generalized power series to solve.

$$y(x) = x^s \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+s}$$

$$\Rightarrow y'(x) = \sum_{n=0}^{\infty} (n+s) a_n x^{n+s-1}$$

$$\Rightarrow y''(x) = \sum_{n=0}^{\infty} (n+s)(n+s-1) a_n x^{n+s-2}$$

Insert this into diff. eq.

$$x^2 y'' + 4xy' + (x^2+2) y = 0$$

$$\Rightarrow x^2 \sum_{n=0}^{\infty} (n+s)(n+s-1) a_n x^{n+s-2}$$

$$+ 4x \sum_{n=0}^{\infty} (n+s) a_n x^{n+s-1}$$

$$+ x^2 \sum_{n=0}^{\infty} a_n x^{n+s} + 2 \sum_{n=0}^{\infty} a_n x^{n+s} = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} (n+s)(n+s-1) a_n x^{n+s}$$

$$+ 4 \sum_{n=0}^{\infty} (n+s) a_n x^{n+s} + \sum_{n=0}^{\infty} a_n x^{n+s+2}$$

$$+ 2 \sum_{n=0}^{\infty} a_n x^{n+s} = 0$$

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re-sum the third term as

$$m = n+2 \; : \; 2 \to \infty$$

$$\Rightarrow n = m-2 \; : \; 0 \to \infty$$

$$\Rightarrow \sum_{n=0}^{\infty} a_n x^{n+s+2} = \sum_{m=2}^{\infty} a_{m-2} x^{m+s}$$

$$= \sum_{n=2}^{\infty} a_{n-2} x^{n+s}$$

Plug back in

$$\Rightarrow \sum_{n=0}^{\infty} (n+s)(n+s-1) a_n x^{n+s}$$

$$+ \sum_{n=0}^{\infty} 4(n+s) a_n x^{n+s} + \sum_{n=2}^{\infty} a_{n-2} x^{n+s}$$

$$+ \sum_{n=0}^{\infty} 2a_n x^{n+s} = 0$$

Can now read off the coefficients of x^{n+s}, but $n=0$ + $n=1$ should be considered individually:

$n=0$: $(n+s)(n+s-1)a_n + 4(n+s)a_n$
$+ 2a_n = 0$

$\Rightarrow \quad s(s-1)a_0 + 4sa_0 + 2a_0 = 0$

$\Rightarrow \quad \left(s(s-1) + 4s + 2\right) a_0 = 0$

For a general solution, $a_0 \neq 0$

$\Rightarrow \quad s(s-1) + 4s + 2 = 0$

$\Rightarrow \quad s^2 - s + 4s + 2 = 0$

$\Rightarrow \quad s^2 + 3s + 2 = 0$

$\Rightarrow \quad (s+1)(s+2) = 0$

$\Rightarrow \quad s = -1$ or $s = -2$

These correspond to two linearly independent solutions.

(analogous to $y'' + y = 0 \Rightarrow y = A\sin x + B\cos x$)

Consider $s=-1$ 1st.

It will generate its own series

$n=1$, $s=-1$ $\Rightarrow$ $(n+s)(n+s-1)a_n + 4(n+s)a_n + 2a_n = 0$

$$\Rightarrow (1-1)(1-1-1)a_1 + 4(1-1)a_1 + 2a_1 = 0$$

$$\Rightarrow a_1 = 0$$

for $n>1$ & $s=-1$

$$\Rightarrow (n-1)(n-2)a_n + 4(n-1)a_n + a_{n-2} + 2a_n = 0$$

$$\Rightarrow \{(n-1)(n-2) + 4(n-1) + 2\}a_n = -a_{n-2}$$

$$\Rightarrow [(n-1)(n+2) + 2]a_n = -a_{n-2}$$

$$\Rightarrow [n^2 + n - 2 + 2]a_n = -a_{n-2}$$

$\Rightarrow \quad n(n+1)\, a_n = -a_{n-2}$

$\Rightarrow$ for $n > 1 \qquad s = -1$

$$a_n = \frac{-a_{n-2}}{n(n+1)}$$

Now set $m = n-2 \Rightarrow n = m+2$

$$\Rightarrow \quad a_{n+2} = \frac{-a_n}{(n+2)(n+3)}$$

Since $a_1 = 0 \Rightarrow a_3 = 0, a_5 = 0, \ldots$ (all odd indices)

$\Rightarrow \quad a_{2m+1} = 0 \quad \forall\, m \in \{0, 1, 2, \ldots\}$

For $n=0$ & even terms, we find that

$$a_2 = \frac{-a_0}{2 \cdot 3}$$

$$a_4 = \frac{-a_2}{4 \cdot 5} = \frac{(-1)^2 a_0}{5 \cdot 4 \cdot 3 \cdot 2}$$

$$a_6 = \frac{-a_4}{6\cdot 7} = \frac{(-1)^3 a_0}{7!}$$

$$\Rightarrow \quad a_{2m} = \frac{(-1)^m a_0}{2m+1!}$$

$\Rightarrow$ for $s = -1$ only even terms

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+s}$$

$$= \sum_{m=0}^{\infty} a_{2m} x^{2m-1}$$

$$= x^{-1} a_0 \sum_{m=0}^{\infty} \frac{(-1)^m}{2m+1!} x^{2m}$$

$$= x^{-2} a_0 \sum_{m=0}^{\infty} \frac{(-1)^m}{2m+1!} x^{2m+1}$$

Recognize

$$\sum_{m=0}^{\infty} \frac{(-1)^m}{2m+1!} x^{2m+1} = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots = \sin x$$

$$\Rightarrow y(x) = a_0 \frac{\sin x}{x^2} \qquad \text{for } s=-1$$

There is a second linearly independent solution for $s=-2$ (hwk problem)

Bessel's equation

- has been studied extensively
- has many applications & arises in situations in which there is cylindrical symmetry

(Legendre equation applies when there is spherical symmetry.)

Bessel's equation is

$$x^2 y'' + xy' + (x^2 - p^2) y = 0 \quad (*)$$

where p is a constant called the order of the Bessel function y, which is the solution of $(*)$.

Consider asymptotic behavior again:

$$|x| \ll 1 \Rightarrow x^2 \lll 1 \simeq 0$$

$$\Rightarrow x y' \approx p^2 y$$

$$\Rightarrow x \frac{dy}{dx} \approx p^2 y$$

$$\Rightarrow \frac{dy}{y} \approx \frac{p^2\, dx}{x}$$

$$\Rightarrow \ln y \approx p^2 \ln x + c$$

$$\Rightarrow y \approx x^{p^2} e^c = x^{p^2} A$$

Since p is unknown, it could be

$p = 1/4 \Rightarrow p^2 = 1/2$ & again

we require generalized power series

Another limit is $|x| \gg 1$

so that $x^2 \gg x$ & $x^2 \gg p^2$

Then

$$x^2 y'' + x^2 y \approx 0$$

$$\Rightarrow y'' + y \approx 0$$

then $y(x) \approx A \sin x + B \cos x$

So then solutions are

sinusoidal in this limit.