Lecture 16

More mathematical properties of Legendre polynomials

- Recall that in 3D, the vectors $\{\hat{i}, \hat{j}, \hat{k}\}$ are a complete orthonormal set.
- We use $\{\hat{i}, \hat{j}, \hat{k}\}$ $= \{\hat{x}_1, \hat{x}_2, \hat{x}_3\} = \{\hat{e}_1, \hat{e}_2, \hat{e}_3\}$ $= \{\hat{x}, \hat{y}, \hat{z}\}$ interchangeably.

orthonormal:

$$\hat{e}_i \cdot \hat{e}_j = \delta_{ij}$$

Complete

$\forall$ vectors $\vec{A} \in \mathbb{R}^3$ $\exists$ scalars $A_1, A_2\ A_3$ such that

$$\vec{A} = A_1 \hat{e}_1 + A_2 \hat{e}_2 + A_3 \hat{e}_3$$

These components may be found via

$$A_1 = \vec{A} \cdot \hat{e}_1$$
$$A_2 = \vec{A} \cdot \hat{e}_2$$
$$A_3 = \vec{A} \cdot \hat{e}_3$$

Observe by plugging in that

$$\vec{A} = \sum_{i=1}^{3} (\vec{A} \cdot \hat{e}_i)\, \hat{e}_i$$
$$= \sum_{i=1}^{3} (\hat{e}_i^T \vec{A})\, \hat{e}_i$$
$$= \sum_{i=1}^{3} \hat{e}_i \cdot \hat{e}_i^T \vec{A}$$
$$= \underbrace{\left(\sum_{i=1}^{3} \hat{e}_i \hat{e}_i^T \right)}\vec{A}$$

It must be the case that this is the identity matrix.

True for any complete orthonormal basis

A set of functions $A_n(x)$ are a complete orthonormal set on an interval $[a,b]$ if

$$\int_a^b A_n^*(x)\, A_m(x)\, dx = \delta_{n,m} \quad \text{(orthonormal)}$$

& any well behaved function $f(x)$ on the interval $[a,b]$ may be written as

$$f(x) = \sum_{n=0}^{\infty} c_n A_n(x)$$

where we demand $\sum_n |c_n|^2 < \infty$ for convergence.

- The infinite-dimensional vector space spanned by $\{A_n(x)\}_{n=0}^{\infty}$ is called a Hilbert space.

Example: From Fourier analysis,

any well behaved function on the interval $[-\pi, \pi]$ may be expanded in terms of the functions

$$A_n(x) = \frac{1}{\sqrt{2\pi}} e^{inx}$$

where $\left\{ \frac{1}{\sqrt{2\pi}} e^{inx} \right\}_{n=0}^{\infty}$ is a complete orthonormal set.

verify orthonormality:

$$\int_{-\pi}^{\pi} \left(\frac{e^{inx}}{\sqrt{2\pi}} \right)^* \left(\frac{e^{imx}}{\sqrt{2\pi}} \right) dx$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-inx} e^{imx} dx$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(m-n)x} dx$$

$$= \frac{1}{2\pi} \begin{cases} 2\pi & : m = n \\ \dfrac{e^{i(m-n)\pi} - e^{-i(m-n)\pi}}{i(m-n)} & : m \neq n \end{cases}$$

Recall that $\dfrac{e^{i\theta} - e^{-i\theta}}{2i} = \sin\theta$

so if $m \neq n$, reduces to

$$\frac{1}{2\pi} \frac{2}{(m-n)} \cdot \frac{e^{i(m-n)\pi} - e^{-i(m-n)\pi}}{2i}$$

$$= \frac{1}{\pi(m-n)} \sin(\pi(m-n))$$

$$= 0$$

$$\Rightarrow \int_{-\pi}^{\pi} \left(\frac{e^{inx}}{\sqrt{2\pi}}\right)^* \left(\frac{e^{imx}}{\sqrt{2\pi}}\right) dx$$

$$= \delta_{n,m}$$

Then we write $f(x) = \sum_{n=-\infty}^{\infty} c_n \frac{e^{inx}}{\sqrt{2\pi}}$

for some Fourier coefficients

– How to find them?

Consider that

$$\int_{-\pi}^{\pi} \left(\frac{e^{imx}}{\sqrt{2\pi}}\right)^* f(x)\, dx$$

$$= \int_{-\pi}^{\pi} \left(\frac{e^{imx}}{\sqrt{2\pi}}\right)^* \sum_{n=-\infty}^{\infty} c_n \frac{e^{inx}}{\sqrt{2\pi}}$$

$$= \sum_{n=-\infty}^{\infty} c_n \left[\int_{-\pi}^{\pi} \left(\frac{e^{imx}}{\sqrt{2\pi}}\right)^* \frac{e^{inx}}{\sqrt{2\pi}}\right]$$

$$= \sum_{n=-\infty}^{\infty} c_n \delta_{n,m}$$

$$= c_m$$

orthogonality of the Legendre polynomials

plan is to show that

$$\int_{-1}^{1} P_\ell(x)\, P_m(x)\, dx = 0 \qquad \text{unless } \ell = m$$

to prove this, consider that Legendre diff. eq. is

$$(1-x^2)\, P_\ell''(x) - 2x\, P_\ell'(x) + \ell(\ell+1)\, P_\ell(x) = 0$$

$$\Rightarrow \quad \frac{d}{dx}\left[(1-x^2)\, P_\ell'(x)\right] + \ell(\ell+1)\, P_\ell(x) = 0$$

Now multiply this by $P_m(x)$.

Do the same but switch ℓ & m & then subtract to get that

$$P_m(x)\,\frac{d}{dx}\left[(1-x^2)\, P_\ell'(x)\right] + \ell(\ell+1)\, P_\ell(x)\, P_m(x)$$

$$-\left(P_\ell(x)\frac{d}{dx}\left[(1-x^2)\, P_m'(x)\right] + m(m+1)\, P_\ell(x)\, P_m(x)\right) = 0$$

$$\Rightarrow P_m(x)\frac{d}{dx}\left[(1-x^2)P_\ell'(x)\right]$$

$$-P_\ell(x)\frac{d}{dx}\left[(1-x^2)P_m'(x)\right]$$

$$+\left[\ell(\ell+1)-m(m+1)\right]P_m(x)P_\ell(x)=0$$

Rewrite first two terms ~~as~~ to get

$$\frac{d}{dx}\left[(1-x^2)\left(P_m(x)P_\ell'(x)-P_\ell(x)P_m'(x)\right)\right]$$

$$+\left[\ell(\ell+1)-m(m+1)\right]P_m(x)P_\ell(x)$$

$$=0$$

Now integrate on $[-1,1]$:

$$(1-x^2)\left(P_m(x)P_\ell'(x)-P_\ell(x)P_m'(x)\right)\Big|_{-1}^{1}$$

$$+\left[\ell(\ell+1)-m(m+1)\right]\int_{-1}^{1}P_m(x)P_\ell(x)\,dx$$

$$=0$$

1st term is equal to zero b/c

$1-x^2=0$ for $+1$ & -1.

Furthermore, when $\ell \neq m$ then

$$\ell(\ell+1) - m(m+1) \neq 0$$

$$\Rightarrow \int_{-1}^{1} P_m(x)\, P_\ell(x)\, dx = 0$$

for $\ell \neq m$.

This method for proving orthogonality is somewhat standard, whenever we have polynomials that are the solutions to differential equations.

Furthermore, any polynomial of degree n can be written as a linear combination of Legendre polynomials of degree $\leq n$. Thus any polynomial of degree $< \ell$ is orthogonal to $P_\ell(x)$, i.e.,

$$\int_{-1}^{1} P_\ell(x)\left[\sum_{j=0}^{\ell-1} c_j x^j\right] dx$$

$$= \int_{-1}^{1} P_\ell(x)\left(\sum_{k=0}^{\ell-1} b_k P_k(x)\right) dx$$

$$= \sum_{k=0}^{\ell-1} \int_{-1}^{1} P_\ell(x)\, P_k(x)\, dx = 0 .$$

Normalization of Legendre polynomials

- Taking the scalar product of a vector w/ itself ~~...~~, $\vec{A}\cdot\vec{A} = A^2$, gives the square of the length of $\vec{A}$.
- Dividing $\vec{A}$ by A gives a unit vector.
- Now we are thinking of functions on an interval as vectors in a vector space.

Can compute their norm as

$$\int_a^b A^*(x)\,A(x)\,dx = \int_a^b |A(x)|^2 dx = N^2$$

Then $\frac{A(x)}{N}$ is normalized,

because its length is 1.

N is called "normalization factor"

For example, $\int_0^\pi \sin^2(nx)\,dx = \pi/2$

$\Rightarrow$ norm of $\sin(nx)$ on $[0,\pi]$ is $\sqrt{\pi/2}$

$\Rightarrow$ $\sqrt{2/\pi}\,\sin(nx)$ have norm 1 on $[0,\pi]$.

Now we prove that

$$\int_{-1}^{1} \left[P_\ell(x)\right]^2 dx = \frac{2}{2\ell+1}$$

$\Rightarrow$ $\left\{P_\ell(x)\cdot\sqrt{\frac{2\ell+1}{2}}\right\}_\ell$ form an orthonormal set on $[-1,1]$.

To prove this, use recursion relation

$$l P_l(x) = x P_l'(x) - P_{l-1}'(x)$$

Now multiply this by $P_l(x)$ & integrate:

$$l \int_{-1}^{1} [P_l(x)]^2 dx$$

$$= \int_{-1}^{1} x P_l(x) P_l'(x) - \int_{-1}^{1} P_l(x) P_{l-1}'(x) dx$$

last integral equal to zero by

$$P_{l-1}(x) = \sum_{j=0}^{l-1} c_j x^j$$

& $P_{l-1}'(x) = \sum_{j=0}^{l-1} c_j x^{j-1} j$

& scalar product of $P_l(x)$ w/ polynomial of degree $l-1$ or less vanishes.

So then

$$\ell \int_{-1}^{1} \left[P_\ell(x)\right]^2 dx = \int_{-1}^{1} x \, P_\ell(x) \, P_\ell'(x) \, dx$$

evaluate RHS using integration by parts; using $\left[P_\ell^2\right]' = 2 P_\ell P_\ell'$

$$\int_{-1}^{1} x \, P_\ell(x) \, P_\ell'(x) \, dx = \frac{x}{2}\left[P_\ell(x)\right]^2 \Big|_{-1}^{1} - \frac{1}{2}\int_{-1}^{1} \left[P_\ell(x)\right]^2 dx$$

$$\boxed{\int f g' \, dx = fg - \int f' g \, dx}$$

$$= 1 - \frac{1}{2}\int_{-1}^{1} \left[P_\ell(x)\right]^2 dx$$

$$\Rightarrow \ell \int_{-1}^{1} \left[P_\ell(x)\right]^2 dx = 1 - \frac{1}{2}\int_{-1}^{1} \left[P_\ell(x)\right]^2 dx$$

$$\Rightarrow (2\ell+1) \int_{-1}^{1} \left[P_\ell(x)\right]^2 dx = 2$$

$$\Rightarrow \int_{-1}^{1} \left[P_\ell(x)\right]^2 dx = \frac{2}{2\ell+1}$$