Lecture 15

Generating function for Legendre polynomials

$$\Phi(x,h) = (1-2xh+h^2)^{-1/2} \quad \text{for } |h|<1$$

This is the generating function for $P_\ell(x)$, which means that expanding $\Phi(x,h)$ in a power series about $h=0$ gives

$$\Phi(x,h) = \sum_{\ell=0}^{\infty} h^\ell P_\ell(x)$$

Bring up Mathematica

Using Series [Φ[x,h], {h, 0, 5}]
we see that

$$\Phi(x,h) = 1\cdot h^0 + x h^1 + \tfrac{1}{2}(3x^2-1)h^2 + \tfrac{1}{2}(5x^3-3x)h^3 + O(h^4)$$

which implies that

$P_0(x) = 1$

$P_1(x) = x$

$P_2(x) = \frac{1}{2}(3x^2-1)$

$P_3(x) = \frac{1}{2}(5x^3-x)$

which agrees w/ first few Legendre polynomials

This does not prove that the coefficient of h^l is $P_l(x)$ $\forall l$, but just the 1st few terms

So we will now show that

$$\Phi(x,h) = \sum_{l=0}^{\infty} h^l p_l(x)$$

gives all coefficients $p_l(x) = P_l(x)$

To do so, we prove that

$$(1-x^2)\,p_l''(x) - 2x p_l'(x) + l(l+1) p_l(x) = 0$$

$\forall \ell$. $\Rightarrow p_\ell(x)\big|_{x=1} = 1 \quad \forall \ell$.

Let us prove the second part 1st.

Recall the geometric series for $|r| < 1$

$$S_N = \sum_{n=0}^{N} r^n = 1 + r + r^2 + \dots + r^N$$

$$\Rightarrow r S_N = r + r^2 + r^3 + \dots + r^{N+1}$$

$$\Rightarrow S_N - r S_N = 1 - r^{N+1}$$

$$\Rightarrow S_N = \frac{1 - r^{N+1}}{1-r}$$

$$\Rightarrow \lim_{N\to\infty} S_N = \sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$$

since

$$\Rightarrow \Phi(1,h) = (1 - 2h + h^2)^{-1/2}$$

$$= \frac{1}{\sqrt{1-2h+h^2}}$$

$$= \frac{1}{\sqrt{(1-h)^2}} = \frac{1}{|1-h|}$$

for $h \in (0,1)$

$$= \frac{1}{1-h} = \sum_{\ell=0}^{\infty} 1 \cdot h^{\ell} = 1 \cdot h^0 + 1 \cdot h^1 + 1 \cdot h^2 + \dots$$

$$= \sum_{\ell=0}^{\infty} P_\ell(1) h^{\ell}$$

then term by term, we get

$$P_\ell(1) = 1 \quad \forall \ell$$

Now consider the following identity:

$$(*) \quad (1-x^2)\frac{\partial^2 \Phi}{\partial x^2} - 2x\frac{\partial \Phi}{\partial x} + h\frac{\partial^2}{\partial h^2}(h\Phi) = 0$$

can verify this easily or use Mathematica.

Now insert $\Phi(x,h) = \sum_{\ell=0}^{\infty} h^{\ell} P_\ell(x)$

& we should then conclude that each $P_\ell(x)$ obeys Legendre's diff. eq.

Consider that

$$\frac{\partial^2 \Phi}{\partial x^2} = \sum_{l=0}^{\infty} h^l \, P_l''(x)$$

$$\frac{\partial \Phi}{\partial x} = \sum_{l=0}^{\infty} h^l \, P_l'(x)$$

$$\begin{aligned}
h \frac{\partial^2 (h\Phi)}{\partial h^2} &= h \frac{\partial^2}{\partial h^2}\left[h \sum_{l=0}^{\infty} h^l \, P_l(x) \right] \\
&= h \frac{\partial^2}{\partial h^2} \sum_{l=0}^{\infty} h^{l+1} P_l(x) \\
&= h \sum_{l=0}^{\infty} \frac{\partial^2}{\partial h^2} h^{l+1} \, P_l(x) \\
&= h \sum_{l=0}^{\infty} (l+1)\, l \, h^{l-1} P_l(x) \\
&= (l+1)\, l \sum_{l=0}^{\infty} h^l \, P_l(x)
\end{aligned}$$

Now insert into (#) & we find that

$$(1-x^2)\sum_{l=0}^{\infty} h^l p_l''(x) - 2x\sum_{l=0}^{\infty} h^l p_l'(x)$$

$$+ \sum_{l=0}^{\infty} h^l (l+1)l\, p_l(x) = 0$$

$$\Rightarrow \sum_{l=0}^{\infty} h^l \Big[(1-x^2)p_l''(x) - 2x\,p_l'(x) + (l+1)\,l\,p_l(x)\Big] = 0$$

So then $p_l(x)$ satisfies Legendre diff. eq. term by term.

& this implies that each $p_l(x)$ on its own & boundary condition $p_l(1)=1$

$$\Rightarrow p_l(x) = P_l(x) \quad \forall l$$

so these are Legendre polynomials

We can also derive recursion relations for the Legendre polynomials.

Recall that

$$\cos(2x) = \cos^2 x - \sin^2 x$$

$$\sin(2x) = 2\sin x \cos x$$

$$\Rightarrow \sin(2mx) = 2\sin(mx)\cos(mx) \quad \forall m \in \{0, 1, 2, \ldots\}$$

$\boxed{\text{Also Fibonacci sequence}}$

Example of recursion relations

& there are recursion relations for Legendre polynomials.

Can easily prove them using the generating function

$$l P_l(x) = (2l-1)\, x P_{l-1} - (l-1) P_{l-2}$$

Numerically, this is the fastest way to generate P_l $\forall l > 1$

E.g., $\ell = 2$

$$2P_2 = (4-1)xP_1 - (2-1)P_0$$

$$= 3x\,P_1 - P_0$$

$$= 3x(x) - (1)$$

$$\Rightarrow P_2(x) = \frac{1}{2}(3x^2 - 1)$$

can then use P_1 & P_2 to generate P_3

<u>Proof</u>: Recall that

$$\Phi(x,h) = (1-2xh+h^2)^{-1/2}$$

$$\Rightarrow \frac{\partial \Phi}{\partial h} = -\frac{1}{2}(1-2xh+h^2)^{-3/2}(-2x+2h)$$

$$= \frac{x-h}{(1-2xh+h^2)^{3/2}}$$

$$= \frac{x-h}{(1-2xh+h^2)\sqrt{1-2xh+h^2}}$$

$$\Rightarrow (1-2xh+h^2)\frac{\partial \Phi}{\partial h} = \frac{1}{\sqrt{1-2xh+h^2}}(x-h)$$

$$\Rightarrow (1-2xh+h^2)\frac{\partial \Phi}{\partial h} = (x-h)\,\Phi(x,h)$$

$$\Rightarrow (1-2xh+h^2)\frac{\partial}{\partial h}\sum_{l=0}^{\infty} h^l P_l(x)$$

$$= (x-h)\sum_{l=0}^{\infty} h^l P_l(x)$$

$$\Rightarrow (1-2xh+h^2)\sum_{l=0}^{\infty} l\cdot h^{l-1} P_l(x)$$

$$= (x-h)\sum_{l=0}^{\infty} h^l P_l(x)$$

$$\Rightarrow \sum_{l=0}^{\infty} l\cdot h^{l-1} P_l(x) + \sum_{l=0}^{\infty}\left[-2x\,l\,h^l P_l(x)\right]$$

$$+ \sum_{l=0}^{\infty} l\cdot h^{l+1} P_l(x)$$

$$= \sum_{l=0}^{\infty} x h^l P_l(x) - \sum_{l=0}^{\infty} h^{l+1} P_l(x)$$

Now relabel some of the series

10

$$\sum_{l=-1}^{\infty} (l+1) h^l P_{l+1}(x) - 2x \sum_{l=0}^{\infty} l h^l \cdot P_l(x)$$

$$+ \sum_{l=1}^{\infty} (l-1) h^l P_{l-1}(x)$$

$$= \sum_{l=0}^{\infty} x h^l P_l(x) - \sum_{l=1}^{\infty} h^l P_{l-1}(x)$$

$$\Rightarrow (l+1) P_{l+1}(x) + [-2xl - x] P_l(x)$$
$$+ (l-1+1) P_{l-1}(x) = 0$$

$$\Rightarrow (l+1) P_{l+1}(x) - x(2l+1) P_l(x) + l P_{l-1}(x) = 0$$

$$\Rightarrow (l+1) P_{l+1}(x) = x(2l+1) P_l(x) + l P_{l-1}(x)$$

to put in form given,

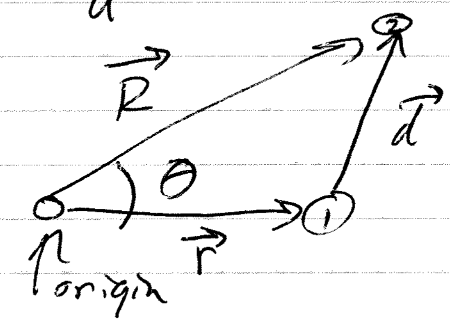
pick $l' = l+1 \Rightarrow l'-1 = l$

$$\Rightarrow l P_l(x) = x(2(l'-1)+1) P_{l-1}(x)$$
$$+ (l-1) P_{l-2}(x)$$
$$= x(2l-1) P_{l-1}(x) + (l-1) P_{l-2}(x)$$

Legendre polynomials show up when there is an inverse square law.

potential energy

$$F = \frac{K}{d^2} \iff U = \frac{K}{d}$$



Suppose two charges or two masses at $\vec{r}$ & $\vec{R}$ w/ $d = |\vec{R} - \vec{r}|$

Electrostatics $\quad U = \dfrac{k q_1 q_2}{d}$

Gravity $\quad U = \dfrac{-G m_1 m_2}{d}$

Apply law of cosines to get that

$$d = |\vec{R} - \vec{r}| = \sqrt{R^2 - 2Rr\cos\theta + r^2}$$

$$= R\sqrt{1 - \frac{2r}{R}\cos\theta + \left(\frac{r}{R}\right)^2}$$

Set $h = \frac{r}{R}$, $x = \cos\theta$

$$\Rightarrow \quad d = R\sqrt{1 - 2hx + h^2}$$ } related to generating function

then

$$U = \frac{K}{d} = \frac{K}{R}\left(1 - 2hx + h^2\right)^{-1/2}$$

$$= \frac{K}{R}\,\Phi(x,h)$$

So that U is proportional to the generating function $\Phi(x,h)$

so then any $\frac{1}{d}$ potential can be expanded in Legendre polynomials

$$U = \frac{K}{R}\sum_{l=0}^{\infty} h^l P_l(x)$$

If $R \gg r$ then $h \ll 1$ and can approximate w/ 1st few terms.

Now suppose a large # of changes q_i @ points $\vec{r}_i$.

Electrostatic potential at $\vec{R}$ due to change q_i @ r_i is given by

$$V_i = K' q_i \sum_{l=0}^{\infty} \frac{r_i^{\,l}}{R^{l+1}} P_l(\cos\theta_i)$$

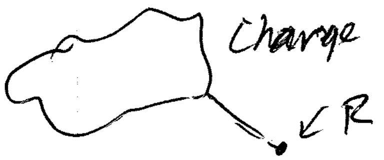
total potential is then

$$V = \sum_i V_i = K' \sum_i q_i \sum_{l=0}^{\infty} \frac{r_i^{\,l}}{R^{l+1}} P_l(\cos\theta_i)$$

$$= K' \sum_{l=0}^{\infty} \sum_i \frac{q_i r_i^{\,l} P_l(\cos\theta_i)}{R^{l+1}}$$

Now consider continuous limit

$$V = \int dV = K' \int_Q dq \sum_{l=0}^{\infty} \frac{r^l P_l(\cos\theta)}{R^{l+1}}$$



$$= K' \int_{Vol} \rho(\vec{r})\, dr^3 \sum_{l=0}^{\infty} \frac{r^l}{R^{l+1}} P_l(\cos\theta)$$

$$= K' \sum_{l=0}^{\infty} \frac{1}{R^{l+1}} \int_{Vol} \rho(\vec{r})\, r^l P_l(\cos\theta)\, dr^3$$

$$dr^3 = r^2 \sin\theta \, dr\, d\theta\, d\varphi$$

Suppose $l=0$, then approximation is

$$= K' \frac{1}{R} \underbrace{\int_{Vol} \rho(\vec{r}) \cdot 1 \cdot 1 \cdot dr^3}_{Q \text{ total charge}}$$

$$= \frac{K'Q}{R}$$ approximate as a monopole

$l=1 \Rightarrow \dfrac{K'}{R^2} \displaystyle\int_{Vol} \rho(\vec{r}) \cdot r \cdot \cos\theta \, dr^3$

$$= \frac{K'}{R^2} \cdot \text{Dipole}$$

etc.