Lecture 14

Recall the binomial expansion:

$$(x+y)^n = \frac{x^n}{0!} y^0 + \frac{n}{1!} x^{n-1} y^1 + \frac{n(n-1)}{2!} x^{n-2} y^2 + \dots + x^0 y^n$$

$$= \binom{n}{0} x^n y^0 + \binom{n}{1} x^{n-1} y^1 + \binom{n}{2} x^{n-2} y^2 + \dots + \binom{n}{n} x^0 y^n$$

$$= \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

where $\binom{n}{k} \equiv \frac{n!}{k!\, n-k!}$

is the binomial coefficient

We can also generate this using Pascal's triangle

1 $\quad n=1$

1 2 1 $\quad n=2$

1 3 3 1 $\quad n=3$

1 4 6 4 1 $\quad n=4$

1 5 10 10 5 1 $\quad n=5$

1 6 15 20 15 6 1 $\quad n=6$

$\vdots$

So that

$$(x+y)^5 = 1\cdot x^5 \cdot y^0 + 5x^4y^1 + 10x^3y^2 + 10x^2y^3 + 5x^1y^4 + 1\cdot x^0y^5$$

These patterns occur also when differentiating products of functions

$$(fg)^{(0)} = f^{(0)}g^{(0)} = fg$$

$$(fg)^{(1)} = (fg)' = f'g + fg' = f^{(1)}g^{(0)} + f^{(0)}g^{(1)}$$

$$(fg)^{(2)} = (fg)'' = f''g + 2f'g' + fg''$$

$$= f^{(2)}g^{(0)} + 2f^{(1)}g^{(1)} + f^{(0)}g^{(2)}$$

$$(fg)^{(3)} = (fg)''' = f'''g + 3f''g' + 3f'g'' + fg'''$$

$$\vdots$$

$$(fg)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(n-k)} \cdot g^{(k)}$$

In operator form, we would write

$$(fg)'' = \frac{d^2}{dx^2}(fg) = \frac{d^2 f}{dx^2}\cdot g + 2\frac{df}{dx}\frac{dg}{dx} + f\frac{d^2 g}{dx^2}$$

$$= \left[\left(\frac{d}{dx}\right)^2 f\right] g + 2\left[\left(\frac{d}{dx}\right)^1 f\right]\left[\left(\frac{d}{dx}\right)^1 g\right] + f\left[\left(\frac{d}{dx}\right)^2 g\right]$$

So then

$$(fg)^{(n)} \equiv \left(\frac{d}{dx}\right)^n (fg)$$

$$= \sum_{k=0}^{n} \binom{n}{k} \left[\left(\frac{d}{dx}\right)^{n-k} f\right]\left[\left(\frac{d}{dx}\right)^{k} g\right]$$

<u>Example:</u>

Find $\left(\frac{d}{dx}\right)^9 \left[x \sin x\right] \equiv (x \sin x)^{'''''''''}$

$n=9$

$$= \sum_{k=0}^{9} \binom{n}{k} \underbrace{\left[\left(\frac{d}{dx}\right)^{n-k} x\right]}_{\text{for } k \in \{0, \dots 7\}} \left[\left(\frac{d}{dx}\right)^{k} \sin x\right]$$

this is equal to zero & then

$$= \binom{9}{8} \frac{d}{dx} x \left[\left(\frac{d}{dx}\right)^{8} \sin x\right] + \binom{9}{9} x \left(\frac{d}{dx}\right)^{9} \sin x$$

Consider that

$$\left(\frac{d}{dx}\right)^n \sin x = \begin{cases} \pm \sin x & n \text{ even} \\ \pm \cos x & n \text{ odd} \end{cases}$$

$+$ n even : n divides by 4

$+$ n odd : $n-1$ divides by 4.

Example: $\left(\frac{d}{dx}\right)^5 (x^5 e^x)$

Since $\left(\frac{d}{dx}\right)^n e^x = e^x$

$$= \sum_{k=0}^{5} \binom{5}{k} \left[\left(\frac{d}{dx}\right)^{5-k} x^5\right] \left[\left(\frac{d}{dx}\right)^k e^x\right]$$

$$= \sum_{k=0}^{5} \binom{5}{k} \left[\left(\frac{d}{dx}\right)^{5-k} x^5\right] e^x$$

$$= \sum_{k=0}^{5} \binom{5}{k} \left[\frac{5!}{5-k!} x^k\right] e^x$$

where we used that

$$\left(\frac{d}{dx}\right)^m x^\ell = \begin{cases} \frac{\ell!}{m!} x^{\ell-m} & \text{for } m \le \ell \\ 0 & \text{for } m > \ell \end{cases}$$

An easy way to generate any Legendre polynomial is via the Rodrigues' formula

$$P_\ell(x) = \frac{1}{2^\ell \ell!} \left(\frac{d}{dx}\right)^\ell (x^2-1)^\ell$$

1st, let us note that

$$v \equiv (x^2-1)^\ell$$

is a polynomial of degree 2ℓ.

If we take ℓ derivatives of it, then we get a polynomial of degree ℓ - that of P_ℓ.

Let us abbreviate $\frac{d^\ell}{dx^\ell} \equiv \partial^\ell$

& so $\partial^\ell v = \frac{d^\ell}{dx^\ell} v = v^{(\ell)}$

to simplify notation.

Proof of Rodrigues formula:

suffices to prove that

$v^{(\ell)}$ satisfies the Legendre

diff. eq.

$$(1-x^2)y'' - 2xy' + \ell(\ell+1)y = 0$$

& that $P_\ell(x) = \left.\frac{v^{(\ell)}(x)}{2^\ell \ell!}\right|_{x=1} = 1$

Consider that

$$\partial v = \frac{d}{dx}(x^2-1)^\ell = \ell(x^2-1)^{\ell-1} 2x$$

$$\Rightarrow (x^2-1)\,\partial v = \ell\,(x^2-1)^{\ell}\,2x$$
$$= 2\ell x v$$

Now differentiate $\ell+1$ times

to get that

$$\partial^{\ell+1}(2\ell x v) = \partial^{\ell+1}\big((x^2-1)\,\partial v\big)$$

where we used above equality

Now apply Leibniz rule

$$\partial^{\ell+1}\left[(x^2-1)\,\partial v\right]$$
$$= \sum_{m=0}^{\ell+1} \binom{\ell+1}{m}\, \partial^m (x^2-1)\, \partial^{\ell+1-m}\, \partial v$$

only $m \in \{0,1,2\}$ give non-zero terms, so that

$$= \binom{\ell+1}{0}\, \partial^0 (x^2-1)\, \partial^{\ell+1} (\partial v) +$$

$$\binom{\ell+1}{1} \partial^1 (x^2-1)\, \partial^\ell(\partial v) \;+$$

$$\binom{\ell+1}{2} \partial^2 (x^2-1)\, \partial^{\ell-1}(\partial v)$$

Note that $\binom{\ell+1}{0} = 1$, $\binom{\ell+1}{1} = \ell+1$,

$$\binom{\ell+1}{2} = \frac{(\ell+1)\ell}{2}$$

$$\Rightarrow\; = (x^2-1)\,\partial^{\ell+2} v + (\ell+1)\, 2x\, \partial^{\ell+1} v$$

$$+ \frac{(\ell+1)\ell}{2} \cdot 2\, \partial^\ell v$$

<u>Now do the RHS:</u>

$$\partial^{\ell+1}[2\ell x v]$$

$$= 2\ell\, \partial^{\ell+1}[x v]$$

$$= 2\ell \sum_{m=0}^{\ell+1} \binom{\ell+1}{m} \partial^m x\, \partial^{\ell+1-m}(v)$$

Note that $\partial^0 x = x$, $\partial^1 x = 1$

$\partial^2 x = 0, \ldots$

So then

$$= 2\ell \left[\binom{\ell+1}{0} x \, \partial^{\ell+1} v + \binom{\ell+1}{1} \partial^{\ell} v \right]$$

$$= 2\ell x \, \partial^{\ell+1} v + 2\ell(\ell+1) \partial^{\ell} v$$

$\Rightarrow$

$$(x^2-1)\partial^{\ell+2} v + (\ell+1) 2x \, \partial^{\ell+1} v$$
$$+ \ell(\ell+1) \partial^{\ell} v =$$
$$2\ell x \, \partial^{\ell+1} v + 2\ell(\ell+1) \partial^{\ell} v$$

$$\Rightarrow (x^2-1)\partial^{\ell+2} v + \left((\ell+1)2x - 2\ell x\right) \partial^{\ell+1} v$$
$$- \ell(\ell+1) \partial^{\ell} v = 0 .$$

$$\Rightarrow (x^2-1)\partial^{\ell+2} v + 2x \, \partial^{\ell+1} v$$
$$- \ell(\ell+1) \partial^{\ell} v = 0$$

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$$\Rightarrow (1-x^2)\frac{\partial^2}{\partial x^2} v^{(\ell)} - 2x\frac{\partial}{\partial x} v^{(\ell)}$$
$$+ \ell(\ell+1) v^{(\ell)} = 0$$

$\Rightarrow v^{(\ell)}$ satisfies Legendre diff. eq.

So does

$$P_\ell(x) \equiv \frac{1}{2^\ell \ell!} \underbrace{\frac{d^\ell}{dx^\ell} v}_{v^{(\ell)}}$$

$$= \frac{1}{2^\ell \ell!} \frac{d^\ell}{dx^\ell} (x^2-1)^\ell$$

What about $P_\ell(x)\Big|_{x=1}$?

Consider that

$$(x^2-1)^\ell = [(x+1)(x-1)]^\ell$$
$$= (x+1)^\ell (x-1)^\ell$$

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$$\Rightarrow P_\ell(x) = \frac{1}{2^\ell \ell!} \partial^\ell \left[(x+1)^\ell (x-1)^\ell \right]$$

$$= \frac{1}{2^\ell \ell!} \sum_{m=0}^{\ell} \binom{\ell}{m} \partial^m (x+1)^\ell \, \partial^{\ell-m} (x-1)^\ell$$

All terms except $m=0$ have an $x-1$ to some power & vanish when $x=1$, so that

$$P_\ell(x)\Big|_{x=1} = \frac{1}{2^\ell \ell!} \binom{\ell}{0} \partial^0 (x+1)^\ell \, \partial^\ell (x-1)^\ell \Big|_{x=1}$$

$$= \frac{1}{2^\ell \ell!} \cdot 1 \cdot (x+1)^\ell \cdot \ell! \, (x-1)^0 \Big|_{x=1}$$

$$= \frac{1}{2^\ell \ell!} 2^\ell \ell! = 1$$

By uniquess of solutions to diff. eq's that have the same boundary conditions, conclude that these are the Legendre polynomials