Lecture 13

Series solution of differential equations

- often physical problems lead to differential equations that need to be solved, and in many cases, partial diff. eq's.
- Here we discuss how to solve them using infinite series.

Begin w/ an example:

Suppose we have

$$y' = 2xy \qquad y' = \frac{dy}{dx}$$

can employ method of series solution

Suppose that the solution has the form of a power series:

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n + \cdots$$

$$= \sum_{n=0}^{\infty} a_n x^n$$

Differentiate term by term to get

$$y' = a_1 + 2a_2 x + 3a_3 x^2 + \dots + n a_n x^{n-1} + \dots$$

$$= \sum_{n=1}^{\infty} n a_n x^{n-1}$$

Substitute this into the original diff. eq.:

$$a_1 + 2a_2 x + 3a_3 x^2 + \dots$$

$$= 2x\left(a_0 + a_1 x + a_2 x^2 + \dots\right)$$

$$= 2a_0 x + 2a_1 x^2 + 2a_2 x^3$$

can also write as

$$\sum_{n=1}^{\infty} n a_n x^{n-1} = 2x\left(\sum_{n=0}^{\infty} a_n x^n\right)$$

$$= \sum_{n=0}^{\infty} 2a_n x^{n+1}$$

Since ~~a~~ given function has a unique series expansion in powers of x, the two series should be identical.

$\Rightarrow a_1 = 0,$

$a_2 = a_0,$

$a_3 = \frac{2}{3} a_1 = 0,$

$a_4 = \frac{1}{2} a_2 = \frac{1}{2} a_0$

can use $\sum_{n=0}^{\infty}$ expression to get general relation.

For LHS, choose $m = n-1$

$\Rightarrow m+1 = n$

$n \in \{1, \ldots, \infty\}$

$\Rightarrow m \in \{0, \ldots, \infty\}$

$$\Rightarrow \sum_{n=1}^{\infty} n \, a_n x^{n-1}$$

$$\Rightarrow \sum_{m=0}^{\infty} (m+1) \, a_{m+1} x^m$$

$$= \sum_{n=0}^{\infty} (n+1) \, a_{n+1} x^n$$

for RHS, set $m = n+1$

$\Rightarrow m-1 = n$

$n \in \{0, \ldots, \infty\}$

$\Rightarrow m \in \{1, \ldots, \infty\}$

$$\Rightarrow \sum_{n=0}^{\infty} 2a_n x^{n+1} = \sum_{m=1}^{\infty} 2\, a_{m-1}\, x^m$$

$$= \sum_{n=1}^{\infty} 2a_{n-1}\, x^n$$

LHS = RHS

$$\Rightarrow \sum_{n=0}^{\infty} (n+1)\, a_{n+1}\, x^n = \sum_{n=1}^{\infty} 2a_{n-1}\, x^n$$

RHS has no $n=0$ term, so then

$n=0 \Rightarrow (0+1)\, a_{0+1} = 0$

$\Rightarrow a_1 = 0$

$\forall$ other terms $n \in \{1, \ldots\}$ we have

$$(n+1)\, a_{n+1} = 2a_{n-1}$$

can use $m = n-1 \Rightarrow m+1 = n$ &

$m \in \{0, \ldots\}$

$$\Rightarrow (m+2)\, a_{m+2} = 2a_m$$

$$\Rightarrow a_m = \begin{cases} 0 & \text{odd } m \\ \frac{2}{m} a_{m-2} & \text{even } m \end{cases}$$

Set $m = 2l$ + this gives

$$a_{2l} = \frac{2}{2l} a_{2l-2} = \frac{1}{l} a_{2l-2}$$

$$= \frac{1}{l}\left[\frac{1}{l-1} a_{2l-4}\right]$$

$$= \frac{1}{l!} a_0$$

$\Rightarrow$ solution is

$$y = a_0 + a_0 x^2 + a_0 \frac{x^4}{2!} + a_0 \frac{x^6}{3!}$$

$$+ \cdots + \frac{1}{m!} a_0 x^{2m} + \cdots$$

$$= a_0 \sum_{m=0}^{\infty} \frac{x^{2m}}{m!} = a_0 e^{x^2}$$

Compare w/ solution we get from a simple method:

$$\frac{dy}{dx} = 2xy$$

use elementary separation of variables

$$\frac{dy}{y} = 2x\,dx$$

Integrating both sides gives

$$\ln y = x^2 + \ln c$$

$$\Rightarrow y = c\,e^{x^2}$$ which is the same as previous w/ $a_0 = c$.

Legendre differential equation

When solving a wave equation in a spherically symmetric potential, like Schrödinger's equation for hydrogen, one arrives at Legendre's diff. eq.

$$(1-x^2)\,y'' - 2xy' + \ell(\ell+1)\,y = 0$$

where ℓ is a constant.

- A way to find the solutions of this equation is to assume a series solution
- Suppose solution is

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

then $y'(x) = \sum_{n=0}^{\infty} a_n n x^{n-1} = \sum_{n=1}^{\infty} a_n n x^{n-1}$

no $n=0$ term

&

$$y''(x) = \sum_{n=0}^{\infty} a_n n(n-1) x^{n-2} = \sum_{n=2}^{\infty} a_n n(n-1) x^{n-2}$$

no $n=0$ or $n=1$ term

insert into original differential equation

$$(1-x^2) \sum_{n=2}^{\infty} a_n n(n-1) x^{n-2}$$

$$- 2x \sum_{n=1}^{\infty} a_n n x^{n-1}$$

$$+ l(l+1) \sum_{n=0}^{\infty} a_n x^n = 0$$

Multiply out terms:

$$\sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} - \sum_{n=2}^{\infty} a_n n(n-1)x^n$$

$$- 2\sum_{n=1}^{\infty} a_n n x^n + \ell(\ell+1)\sum_{n=0}^{\infty} a_n x^n = 0$$

only resum the 1st term

pick $m = n-2$

$\Rightarrow m+2 = n$

$n \in \{2, \dots\}$

$\Rightarrow m \in \{0, \dots\}$

$$\Rightarrow \sum_{n=2}^{\infty} a_n (n(n-1)) x^{n-2}$$

$$= \sum_{m=0}^{\infty} a_{m+2}(m+2)(m+1)x^m$$

$$= \sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)x^n$$

Then the above becomes

$$\sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)x^n - \sum_{n=2}^{\infty} a_n n(n-1)x^n$$

$$-2\sum_{n=1}^{\infty} a_n \cdot n x^n + \ell(\ell+1)\sum_{n=0}^{\infty} a_n x^n = 0$$

consider $n=0$ & $n=1$ cases as special

$n=0$

$$a_2(2)(1) - 0 - 0 + \ell(\ell+1)\, a_0 = 0$$

$$\Rightarrow\ 2a_2 + \ell(\ell+1)\, a_0 = 0$$

$$\Rightarrow\ a_2 = -\frac{\ell(\ell+1)}{2}\, a_0$$

$n=1$

$$a_3(3)(2) - 0 - 2a_1 + \ell(\ell+1)\, a_1 = 0$$

$$\Rightarrow\ a_3 = \frac{-\ell(\ell+1)+2}{6}\, a_1$$

even & odd behave differently.

For $n>1$:

$$a_{n+2}(n+2)(n+1) \cancel{+}\, a_n(n(n-1))$$

$$-2a_n \cdot n + \ell(\ell+1)\, a_n = 0$$

$$\Rightarrow\ a_{n+2} = \frac{-\ell(\ell+1) + n^2 + n}{(n+2)(n+1)}\, a_n$$

can rewrite $-\ell(\ell+1)+n^2+n$

$$= (n-\ell)(\ell+n+1)$$

$$\Rightarrow \quad a_{n+2} = \frac{(n-\ell)(\ell+n+1)}{(n+2)(n+1)} a_n$$

get even terms by starting from a_0

$$a_2 = \frac{(0-\ell)(\ell+0+1)}{(0+2)(0+1)} a_0$$

$$= \frac{-\ell(\ell+1)}{2\cdot 1} a_0$$

$$a_4 = \frac{(2-\ell)(\ell+2+1)}{(2+2)(2+1)} a_2$$

$$= \frac{(2-\ell)(\ell+3)}{4\cdot 3} a_2$$

$$= \frac{(\ell-2)(\ell+3)\,\ell(\ell+1)}{4\cdot 3\cdot 2\cdot 1} a_0$$

Then a_n for n even is given by

$$a_n = \frac{\ell(\ell+1)(\ell-2)(\ell+3)\cdots(\ell+n-1)}{n!}(-1)^{n/2}$$

for odd n:

$n=3$ $$a_5 = \frac{(3-\ell)(\ell+3+1)}{(3+2)(3+1)}a_3$$

$$= \frac{(3-\ell)(\ell+4)}{5\cdot 4}a_3$$

$$= \frac{-(\ell-3)(\ell+4)}{5\cdot 4}\left[\frac{-\ell(\ell+1)+2}{6}\right]a_1$$

$$= \frac{(\ell-3)(\ell+4)\left(\ell(\ell+1)-2\right)}{5\cdot 4\cdot 3\cdot 2}a_1$$

$$= \frac{(\ell-3)(\ell+4)(\ell-1)(\ell+2)}{5!}a_1$$

general pattern for n odd is

$$a_n = \frac{(\ell-1)(\ell+2)(\ell-3)(\ell+4)\cdots(\ell+n+1)(-1)^{\frac{n-}{2}}}{n!}$$

general solution is then

$$y = a_0\left[1 - \frac{\ell(\ell+1)}{2!}x^2 + \frac{\ell(\ell+1)(\ell-2)(\ell+3)}{4!}x^4 \cdots\right.$$

$$+ a_1\left\{x - \frac{(\ell-1)(\ell+2)}{3!}x^3\right.$$

$$+ \frac{(\ell-1)(\ell+2)(\ell-3)(\ell+4)}{5!}x^5 -$$

$$= a_0 \sum_{m=0}^{\infty} (-1)^m \frac{\ell(\ell+1)(\ell-2)(\ell+3)\cdots(\ell+2m-1)}{2m!} x^{2m}$$

$$+ a_1 \sum_{m=1}^{\infty} (-1)^m \frac{(\ell-1)(\ell+2)(\ell-3)(\ell+4)\cdots(\ell+2m+2)}{2m+1!} x^{2m+}$$

can use ratio test to
show that series converges
for $x^2 < 1$

Legendre polynomials

for $l=0$ & $x^2=1$,
the a_1 series is $1+\frac{1}{3}+\frac{1}{5}+\cdots$
which is divergent.
In applications, x is the
cosine of an angle θ &
so we would like to have
convergence. How to get it?

for $l=0$,
the a_0 series gives $y=a_0$ since
all other terms have l & thus
are equal to zero (series truncates)

for $l=1$, a_0 series is divergent @ $x^2=1$

but a_1 series stops w/ $y=a_1 x$

For any integer l, one series terminates, giving a polynomial solution, while the other series is divergent @ $x^2=1$

$\Rightarrow$ a set of solutions (polynomial), one for each non-negative integer l.

If we choose a_0 or a_1 such that $y=1$ when $x=1$, then resulting polynomials are called Legendre polynomials, written as $P_l(x)$.

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1st few Legendre polynomials are

$P_0(x) = 1$

$P_1(x) = x$

$P_2(x) = \frac{1}{2}(3x^2 - 1)$

$P_3(x) = \frac{1}{2}(5x^3 - 3x)$

$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$

etc.