Lecture 12

Elliptic integrals

- In the 1700s, Legendre posed the question: "How far does the earth travel in one year?"
- If the Earth's orbit were exactly circular, then the answer is $2\pi R$ w/ $R = 1$ astronomical unit
- The distance between Jan 1 + Apr. 1 is

$$\frac{1}{4} 2\pi R = \frac{\pi R}{2}$$

- In general, we have φR + the velocity is $\frac{ds}{dt} = \dot{\varphi} R = \omega R$ where $\omega = \frac{2\pi}{T}$ w/ $T = 1$ year

- However, from Kepler's laws, we know that the orbit is elliptical.
- For Earth, ellipse is approximately circular, but for long period comets, they are highly eccentric.

- The equation for an ellipse w/ major axis $a > b$ = minor is

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

This reduces to equation of circle for $a = b = R$.

- $\epsilon \equiv \sqrt{\frac{a^2 - b^2}{a^2}}$ is eccentricity of ellipse.

$\epsilon = 0$ iff $a = R = b$ for circle

We can also write the equation of the ellipse in parametric form as

$$x = a\sin\theta, \quad y = b\cos\theta$$

for $a > b$. Then for $a > b$, we have that (for arclength around ellipse)

$$ds^2 = dx^2 + dy^2$$

$$= [a\cos\theta \, d\theta]^2 + [-b\sin\theta \, d\theta]^2$$

$$= \left[a^2\cos^2\theta + b^2\sin^2\theta\right] d\theta^2$$

$$= \left[a^2(1-\sin^2\theta) + b^2\sin^2\theta\right] d\theta^2$$

$$= \left[a^2 - (a^2-b^2)\sin^2\theta\right] d\theta^2$$

$$= a^2\left[1 - \epsilon^2\sin^2\theta\right] d\theta^2$$

$$\Rightarrow \quad ds = a\sqrt{1-\epsilon^2\sin^2\theta}\, d\theta$$

Then to get arclength from $0 \to \varphi$, just integrate from $\theta = 0$ to $\theta = \varphi$

$$a \int_0^{\varphi} \sqrt{1 - \epsilon^2 \sin^2\theta}\, d\theta$$

only if $\epsilon = 0$ does this reduce to

$$a \int_0^{\varphi} d\theta = a\varphi = R\varphi \qquad \text{(for circle)}$$

Another special case is $\epsilon = 1$

$$\Rightarrow \quad a \int_0^{\varphi} \sqrt{1 - \sin^2\theta}\, d\theta$$

$$= a \int_0^{\varphi} \cos\theta \, d\theta = a \sin\varphi .$$

For arbitrary ε, there is no antiderivative of $\sqrt{1-\epsilon^2 \sin^2\theta}$. So just like log, erf, Γ, & B, we define elliptic integrals of the 1st & 2nd kind as

(1st) $F(\varphi, k) = \int_0^{\varphi} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} \quad 0 \le k \le 1$

(2nd) $E(\varphi, k) = \int_0^{\varphi} \sqrt{1-k^2\sin^2\theta}\, d\theta \quad 0 \le k \le 1$

- So the orbit problem reduces to ~~finding~~ solving an elliptic integral of the second kind $\epsilon^2 = k^2$

- Earth has $\epsilon = 0.167$
  $a = 1.0$ AU

- Halley's comet has $\epsilon = 0.967$
  $a = 18.09$ AU

In one orbit, Earth travels

$$a \int_0^{2\pi} \sqrt{1-\epsilon^2\sin^2\theta}\, d\theta$$

can evaluate in Mathematica using

a· EllipticE [φ, k]

So

$$S_{EARTH} = N[a * \text{EllipticE}[2\pi, \epsilon]]$$

$$= 6.012 \text{ AU}$$

compare w/ circular approximation

$$2\pi R \approx 6.283 \text{ AU}$$

which is astronomically off!

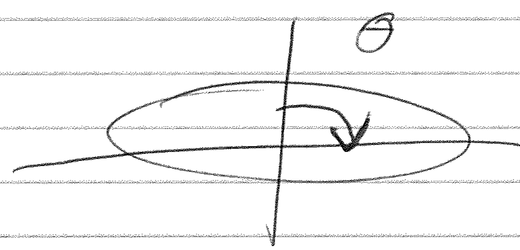
(by about .271 AU, $\approx 1/3$ distance to the sun)

$S_{Halley} = 75.48$ AU,

& compare to $2\pi R = 113.663$,

off by 38 times EARTH-SUN distance.

- $F(\varphi,k)$ & $E(\varphi,k)$ as we defined previously are typically called incomplete elliptic integrals
- Complete elliptic integrals correspond to 1/4 of ellipse



$$K(k) \equiv F(\pi/2, k)$$

$$E(k) \equiv E(\pi/2, k)$$

"Jacobi form" is found by substitution $t = \sin\theta$

$$F(x,k) = \int_0^x \frac{dt}{\sqrt{1-t^2}\sqrt{1-k^2t^2}}$$

$$E(x,k) = \int_0^x \frac{\sqrt{1-k^2t^2}}{\sqrt{1-t^2}}\,dt$$

Proof: $t = \sin\theta$

$$dt = \cos\theta \, d\theta$$
$$= \sqrt{\cos^2\theta} \, d\theta$$
$$= \sqrt{1-\sin^2\theta} \, d\theta$$
$$= \sqrt{1-t^2} \, d\theta$$
$$\Rightarrow d\theta = \frac{dt}{\sqrt{1-t^2}}$$

$\theta : 0 \rightarrow \varphi \Rightarrow t : \sin 0 = 0 \rightarrow \sin\varphi = x$

Complete Jacobi

$$K(k) = F(1,k)$$

$$E(k = E(1,k)$$

Example 1

Evaluate $I = \int_0^{\pi/3} \sqrt{1 - \frac{1}{2}\sin^2\theta}\, d\theta$

take $k^2 = 1/2 \Rightarrow k = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

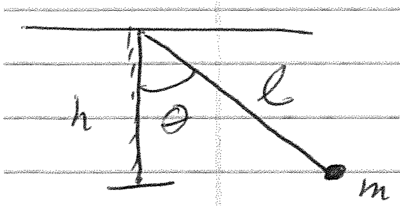
$\varphi = \pi/3$

$I = N\{\text{EllipticE}\{\pi/3, \frac{\sqrt{2}}{2}\}\} \cong 0.96$

Return to pendulum problem

Previously we derived that the equation of motion is

$$\dot{\theta}^2 = \frac{2g}{l}\cos\theta + \text{const.}$$



Now consider swings starting from any angle, e.g. α,

$\Rightarrow \dot{\theta} = 0$ when $\theta = \alpha$

then $\quad -\frac{2g}{l}\cos\alpha = +\text{const.}$

$\Rightarrow$ ~~cos~~

$$\dot{\theta}^2 = \frac{2g}{\ell}(\cos\theta - \cos\alpha)$$

$$\Rightarrow \quad \dot{\theta} = \sqrt{\frac{2g}{\ell}(\cos\theta - \cos\alpha)}$$

Integrating gives

$$\int_0^\alpha \frac{d\theta}{\sqrt{\cos\theta - \cos\alpha}} = \sqrt{\frac{2g}{\ell}} \int_0^{T_\alpha/4} dt$$

$$= \sqrt{\frac{2g}{\ell}}\, T_\alpha/4$$

where T_α is period for swings from $-\alpha$ to α & back.

Use some trig identities to rewrite this as an elliptic integral.

$$\cos\theta = 1 - 2\sin^2(\theta/2)$$

$$\cos\alpha = 1 - 2\sin^2(\alpha/2)$$

$$\Rightarrow T_a = 4\sqrt{\frac{l}{2g}}\int_0^\alpha \frac{d\theta}{\sqrt{(1-2\sin^2(\theta/2)) - [1-2\sin^2(\alpha/2)]}}$$

$$= 4\sqrt{\frac{l}{2g}}\int_0^\alpha \frac{d\theta}{\sqrt{2}\sqrt{\sin^2(\alpha/2) - \sin^2(\theta/2)}}$$

$$= 2\sqrt{\frac{l}{g}}\cdot\frac{1}{\sin\alpha/2}\int_0^\alpha \frac{d\theta}{\sqrt{1-\left(\frac{\sin\theta/2}{\sin\alpha/2}\right)^2}}$$

can now recognize as elliptic integral of 1st kind. can further reduce to see that it is complete elliptic integral.

take $x = \dfrac{\sin\theta/2}{\sin\alpha/2}$

$$dx = \frac{\frac{1}{2}\cos\theta/2}{\sin\alpha/2}\,d\theta$$

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$$\Rightarrow d\theta = \frac{2\sin \alpha/2}{\cos\theta/2}\,dx$$

$$= \frac{2\sin\alpha/2}{\sqrt{1-\sin^2\theta/2}}\,dx$$

$$= \frac{2\sin\alpha/2}{\sqrt{1-\sin^2\alpha/2\left(\frac{\sin^2\theta/2}{\sin^2\alpha/2}\right)}}\,dx$$

$$= \frac{2\sin\alpha/2}{\sqrt{1-\sin^2\alpha/2\ x^2}}\,dx$$

$$= \frac{2k}{\sqrt{1-k^2x^2}}\,dx \quad w/\ k=\sin\alpha/2$$

$$\Rightarrow T_2 = 2\sqrt{\frac{l}{g}}\,\frac{1}{k}\cdot\int_0^1 \frac{2k}{\sqrt{1-k^2x^2}\cdot\sqrt{1-x^2}}\,dx$$

$$= 4\sqrt{\frac{l}{g}}\int_0^1 \frac{1}{\sqrt{1-k^2x^2}\cdot\sqrt{1-x^2}}\,dx$$

This is complete elliptic integral of 1st kind in Jacobi form:

$$K(k) = F(\pi/2, k) \quad w/\ k = \sin \alpha/2$$

$$\Rightarrow T_\alpha = 4\sqrt{\frac{l}{g}}\, K\left[\sin \alpha/2\right]$$

To approximate, let's expand about $k = \sin\alpha/2$ using Mathematica

$$K[k] = \pi/2\left[1 + \left(\frac{1}{2}\right)^2 k^2 + \left(\frac{1\cdot 3}{2\cdot 4}\right) k^4 + \ldots\right]$$

If $\alpha \ll 1$, then $k = \sin\frac{\alpha}{2}$

$$\approx \frac{\alpha}{2} \ll 1$$

$$\Rightarrow T_\alpha = 4\sqrt{\frac{l}{g}}\, \pi/2\left[1 + \frac{1}{4}\left(\frac{\alpha}{2}\right)^2\right]$$

$$= 2\pi\sqrt{\frac{l}{g}}\left[1 + \frac{\alpha^2}{16}\right]$$