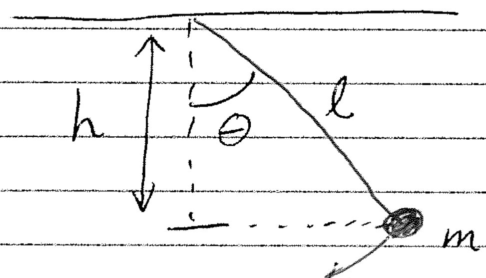
Lecture 10

Solving the simple pendulum in terms of the Beta function



point mass m @ end of massless rod/string of length l

Kinetic energy is given by $\frac{1}{2}mv^2$

w/ $v = \frac{ds}{dt} = \frac{d[l\theta]}{dt} = l\dot{\theta}$

$\Rightarrow T = \frac{1}{2}mv^2 = \frac{1}{2}ml^2\dot{\theta}^2$

$V = -mgh$ = potential energy (= 0 when rod is horizontal)

$= -mgl\cos\theta$

Lagrangian is

$$L = T - V = \frac{1}{2} m l^2 \dot{\theta}^2 + mgl\cos\theta$$

Lagrange equation of motion is then

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = 0$$

$$\Rightarrow \frac{d[ml^2\dot{\theta}]}{dt} + mgl\sin\theta = 0$$

$$\Rightarrow ml^2\ddot{\theta} + mgl\sin\theta = 0$$

$$\Rightarrow \ddot{\theta} = -\frac{g}{l}\sin\theta$$

typical approximation to make is the small angle approximation so that $\theta \approx \sin\theta$

& then $\ddot{\theta} = -\frac{g}{l}\theta$

would then guess a solution of the form $\theta = e^{\alpha t} \Rightarrow \ddot{\theta} = \alpha^2 e^{\alpha t}$

$$\Rightarrow \ddot{\theta} + \frac{g}{l}\theta = 0$$

$$\Rightarrow \alpha^2 e^{\alpha t} + \frac{g}{l} e^{\alpha t} = 0$$

$$\Rightarrow \alpha^2 = -\frac{g}{l}$$

$$\Rightarrow \alpha = \pm i\sqrt{\frac{g}{l}} = \pm i\omega$$

$\Rightarrow$ general solution of

$$\theta(t) = A e^{i\sqrt{\frac{g}{l}}t} + B e^{-i\sqrt{\frac{g}{l}}t}$$

$$= A' \cos \omega t + B' \sin \omega t$$

$$\text{Period} = T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{l}{g}}$$

Now, what if it does not hold that

$|\theta| \ll 1$?

Solve even for large θ

diff. eq. is $\ddot{\theta} = -\frac{g}{\ell} \sin\theta$

Multiply both sides by $\dot{\theta}$

$$\dot{\theta}\ddot{\theta} = -\frac{g}{\ell}\dot{\theta}\sin\theta$$

Consider that $\frac{1}{2}\frac{d}{dt}[\dot{\theta}]^2 = [\dot{\theta}]\ddot{\theta}$

& $\frac{d}{dt}\cos\theta = -\sin\theta\,\dot{\theta}$

$$\Rightarrow \frac{1}{2}\frac{d}{dt}[\dot{\theta}]^2 = \frac{g}{\ell}\frac{d}{dt}\cos\theta$$

Now integrate to find that

$\frac{1}{2}\dot{\theta}^2 = \frac{g}{\ell}\cos\theta + C$ for some constant C.

return to general solution later.

For now, suppose that

$\theta(t=0) = \pi/2$ (90° initial angle)

Initial velocity is $= 0$

& $\cos\theta = \cos^0 = 0$

$\Rightarrow$ $C = 0$ for this case

so that

$$\frac{1}{2}\dot{\theta}^2 = \frac{g}{\ell}\cos\theta$$

$$\Rightarrow \quad \dot{\theta} = \sqrt{\frac{2g}{\ell}}\sqrt{\cos\theta}$$

$$\Rightarrow \quad \frac{d\theta}{dt} = \sqrt{\frac{2g}{\ell}}\sqrt{\cos\theta}$$

$$\Rightarrow \quad \frac{d\theta}{\sqrt{\cos\theta}} = \sqrt{\frac{2g}{\ell}}\,dt$$

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If t goes from 0 to $T/4$ (T period)

then θ goes from 0 to $\pi/2$

$\Rightarrow$ after integrating

$$\int_0^{\pi/2} \frac{d\theta}{\sqrt{\cos\theta}} = \frac{T}{4}\sqrt{\frac{2g}{\ell}}$$

$$\Rightarrow \quad T = 4\sqrt{\frac{\ell}{2g}} \int_0^{\pi/2} \frac{d\theta}{\sqrt{\cos\theta}}$$

Recall that Beta function can be written as

$$\frac{1}{2} B(p,q) = \int_0^{\pi/2} [\cos\theta]^{2q-1} [\sin\theta]^{2p-1} d\theta$$

Take $2p-1 = 0$ & $2q-1 = -1/2$

$\Rightarrow \quad p = 1/2$ & $q = 1/4$

$$\Rightarrow \quad \frac{1}{2} B[1/2, 1/4] = \int_0^{\pi/2} \frac{d\theta}{\sqrt{\cos\theta}}$$

$$\Rightarrow \quad T = 4\sqrt{\frac{l}{2g}}\ \frac{1}{2}\ B[1/2,\ 1/4]$$

$$= \sqrt{\frac{2l}{g}}\ B[1/2,\ 1/4]$$

Taking a numerical approximation in Mathematica gives

$$\sqrt{2}\ B[1/2,\ 1/4] \approx 7.4163$$

$$\Rightarrow T = 7.4163\sqrt{\frac{l}{g}}$$

Compare to

$$T = 2\pi\sqrt{\frac{l}{g}} \approx 6.28\sqrt{\frac{l}{g}}$$

for small angle approximation

longer period is due to "hang time" @ ~~ ~~ $T = 0$.

Move on to Gaussian error function

encounter this in probability theory

bell-shaped curve is $y = e^{-x^2}$ &
error function is area under part of
this curve:

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt$$

Other closely related functions are
cumulative Gaussian distribution:

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-t^2/2}\, dt$$

$$= \frac{1}{2} + \frac{1}{2}\,\mathrm{erf}\left(x/\sqrt{2}\right)$$

Also, $\mathrm{erf}(x) = 2\,\Phi(\sqrt{2}\,x) - 1$

There is also the complementary error function:

$$\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} dt = 1 - \operatorname{erf}(x)$$

Some facts about error function

It is odd: $\operatorname{erf}(-x) = -\operatorname{erf}(x)$

$\operatorname{erf}(\infty) = 1$ b/c

$$\operatorname{erf}(\infty) = \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-t^2} dt = \frac{2}{\sqrt{\pi}} \frac{1}{2} \Gamma(1/2)$$

$$= \frac{2}{\sqrt{\pi}} \frac{1}{2} \sqrt{\pi} = 1$$

For small values of x (i.e., $|x| \ll 1$) we have by Taylor expansion

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt = \frac{2}{\sqrt{\pi}} \int_0^x \left(1 - t^2 + \frac{t^4}{2!} - \dots\right) dt$$

$$= \frac{2}{\sqrt{\pi}}\left(x - \frac{x^3}{3} + \frac{x^5}{5\cdot 2!} - \dots\right)$$

For large x, $x > 3$, erf(x) differs from $\mathrm{erf}(\infty) = 1$ by $\leq 10^{-4}$

Then more interested in $1 - \mathrm{erf}(x)$
$= \mathrm{erfc}(x)$

the function $\mathrm{erfi}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{t^2}\,dt$

can show that

$$\mathrm{erf}(ix) = i \cdot \mathrm{erfi}(x)$$