Lecture 9

Special functions

Integrals, series, & functions of this chapter arise in a variety of physical problems.

Example: Suppose you want to know

$$\int_0^{\pi/2} d\theta \, \frac{1}{\sqrt{\cos\theta}} .$$

One computer program might give $\sqrt{2}\, K(1/\sqrt{2})$ & another gives

$$2\sqrt{\pi}\; \Gamma(5/4) \,/\, \Gamma(3/4)$$

Books might give $\frac{1}{2} B(1/4, 1/2)$ &

$$\{\Gamma(1/4)\}^2 / \sqrt{8\pi} .$$

Which one is correct? They all are purpose of this chapter is to show this.

- In thermal physics, there is the approximation $\ln N! \approx N \ln N - N$. Later we will discuss this approximation + it accuracy.

Factorial function

For $\alpha > 0$, consider that

$$\int_0^\infty e^{-\alpha x}\,dx = -\frac{1}{\alpha} e^{-\alpha x}\Big|_0^\infty = \frac{1}{\alpha}$$

Now differentiate both sides wrt α

$$\frac{d}{d\alpha}\left[\int_0^\infty e^{-\alpha x}\,dx\right] = \int_0^{+\infty} -x e^{-\alpha x}\,dx$$

$$= -\frac{1}{\alpha^2}$$

$$\Rightarrow \int_0^\infty x e^{-\alpha x}\,dx = \frac{1}{\alpha^2}$$

Again

$$\int_0^\infty x^2 e^{-\alpha x} dx = \frac{2}{\alpha^3}$$

& again

$$\int_0^\infty x^3 e^{-\alpha x} dx = \frac{3!}{\alpha^4}$$

In general:

$$\int_0^\infty x^n e^{-\alpha x} = \frac{n!}{\alpha^{n+1}}$$ for integer $n \geq 1$.

Setting $\alpha = 1$ gives

$$\int_0^\infty x^n e^{-x} = n!$$

Thus, we have obtained a definite integral for which the value is $n!$ for integer $n \geq 1$.

We can even use this to give a consistent meaning to $0!$

Setting $n=0$ on LHS gives

$$\int_0^\infty e^{-x} = 1$$

Gamma function (Γ)

Can define the factorial function for non-integral n as

$$\int_0^\infty x^n e^{-x}\,dx = n!$$

Sometimes notation $n!$ for non-integral n means this.

However, common practice is to use Γ notation for Γ function, defined as

$$\Gamma(p) = \int_0^\infty x^{p-1} e^{-x}\,dx \quad p>0$$

For $p \leq 0$ integral diverges & is undefined converges for $p > 0$.

Then $\Gamma(n) = \int_0^\infty x^{n-1} e^{-x} dx$

$= (n-1)!$

$\Gamma(n+1) = \int_0^\infty x^n e^{-x} dx$

$= n!$

Then $\Gamma(1) = 0! = 1$

$\Gamma(2) = 1! = 1$

$\Gamma(3) = 2! = 2$

$\Gamma(4) = 3! = 6$.

Integrate $\Gamma(p+1) = \int_0^\infty x^p e^{-x} dx$ by parts.

Set $x^p = u$, $e^{-x} dx = dv$

then

$$du = p\,x^{p-1}dx \quad , \quad v = -e^{-x}$$

$$\Rightarrow \Gamma(p+1) =$$

$$\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx$$

$$= x^p(-e^{-x})\Big|_0^\infty - \int_0^\infty (-e^{-x})\,p\,x^{p-1}dx$$

$$= p\int_0^\infty e^{-x}x^{p-1}dx$$

$$= p\,\Gamma(p)$$

$$\Rightarrow \quad \Gamma(p+1) = p\,\Gamma(p)$$ recursion relation for Γ function

can be used to simplify expressions involving Γ.

For example:

$$\frac{\Gamma(1/4)}{\Gamma(9/4)} = \frac{\Gamma(1/4)}{5/4\,\Gamma(5/4)} = \frac{\Gamma(1/4)}{(5/4)\cdot(1/4)\,\Gamma(1/4)} = \frac{16}{5}$$

Previously, we did not define Γ for negative numbers, but we can reverse this recursion relation to do so:

$$\Gamma(p) = \frac{1}{p}\Gamma(p+1)$$

defines $\Gamma(p)$ for $p < 0$.

For example:

$$\Gamma(-0.3) = \frac{1}{-0.3}\,\Gamma(0.7)$$

$$\Gamma(-1.3) = \frac{1}{(-1.3)(-0.3)}\,\Gamma(0.7)$$

etc.

Since $\Gamma(1) = 1$, then

$$\Gamma(p) = \frac{\Gamma(p+1)}{p} \to \infty \text{ as } p \to 0$$

this ~~f~~ recursion relation implies that

$\Gamma(p) = \infty$ not just @ zero but @ every negative integer

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evaluating formulas involving Γ functions

$$\Gamma(1/2) = \int_0^\infty \frac{1}{\sqrt{t}} e^{-t}\, dt$$

Set $t = y^2$ & then $dt = 2y\,dy$

$$\Rightarrow \Gamma(1/2) = \int_0^\infty \frac{1}{y} e^{-y^2} 2y\, dy$$

$$= 2\int_0^\infty e^{-y^2} dy$$

can alternatively write as

$$\Gamma(1/2) = 2\int_0^\infty e^{-x^2} dx$$

then

$$\left[\Gamma(1/2)\right]^2 = 4\int_0^\infty\!\!\int_0^\infty e^{-(x^2+y^2)}\, dx\, dy$$

Integral is over 1st quadrant & more easily evaluated using polar coordinates:

$$\left[\Gamma(1/2)\right]^2 = 4\int_0^{\pi/2}\int_0^{\infty} e^{-r^2}\, r\,dr\,d\theta$$

$$= 4\cdot\pi/2 \int_0^{\infty} e^{-r^2} r\,dr$$

$$= 4\cdot\pi/2 \left[\frac{e^{-r^2}}{-2}\Big|_0^{\infty}\right]$$

$$= \pi$$

$$\Rightarrow \boxed{\Gamma(1/2) = \sqrt{\pi}}$$

Another important formula involving Γ functions is

$$\Gamma(p)\,\Gamma(1-p) = \frac{\pi}{\sin \pi p}$$

Beta Function

$$B(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}\,dx \quad p>0, \; q>0.$$

Simple transformations:

$$B(p,q) = B(q,p)$$

simply b/c integration is over $[0,1]$ & symmetry of $x \leftrightarrow 1-x$

can change range of integration to

$x = y/a$ & then $x=1 \Leftrightarrow y=a$

$$\Rightarrow B(p,q) = \int_0^a \left(\frac{y}{a}\right)^{p-1}\left(1-\frac{y}{a}\right)^{q-1}\frac{dy}{a}$$

$$= \frac{1}{a^{p+q-1}}\int_0^a y^{p-1}(a-y)^{q-1}\,dy$$

can also obtain trigonometric form

Set $x = \sin^2\theta$ then

$$dx = 2\sin\theta\cos\theta\, d\theta$$

$$1-x = 1-\sin^2\theta = \cos^2\theta \text{ \&}$$

$$x=1 \Leftrightarrow \theta = \pi/2$$

$$\Rightarrow B(p,q) = \int_0^{\pi/2} (\sin^2\theta)^{p-1}(\cos^2\theta)^{q-1} \cdot 2\sin\theta\cos\theta\, d\theta$$

$$= 2\int_0^{\pi/2} (\sin\theta)^{2p-1}(\cos\theta)^{2q-1}\, d\theta$$

Also, if we substitute $x = y/1+y$

then

$$B(p,q) = \int_0^\infty \frac{y^{p-1}}{(1+y)^{p+q}}\, dy$$

Can write Beta function in terms of

Γ function:

$$B(p,q) = \frac{\Gamma(p)\,\Gamma(q)}{\Gamma(p+q)}$$

To prove this, start w/

$$\Gamma(p) = \int_0^\infty t^{p-1} e^{-t}\,dt$$

& set $t = y^2$

$$\Rightarrow \Gamma(p) = 2\int_0^\infty y^{2p-1} e^{-y^2}\,dy$$

Similarly, $\Gamma(q) = 2\int_0^\infty x^{2q-1} e^{-x^2}\,dx$

Multiply to get

$$\Gamma(p)\,\Gamma(q) = 4\int_0^\infty\int_0^\infty x^{2q-1} y^{2p-1} e^{-(x^2+y^2)}\,dx\,dy$$

convert to polar:

$$= 4 \int_0^\infty \int_0^{\pi/2} (r\cos\theta)^{2q-1} (r\sin\theta)^{2p-1} e^{-r^2} \, r\,dr \, d\theta$$

$$= 4 \int_0^\infty r^{2p+2q-1} e^{-r^2} dr \cdot \int_0^{\pi/2} (\cos\theta)^{2q-1} \cdot (\sin\theta)^{2p-1} d\theta$$

$$= 4 \left[\frac{1}{2} \Gamma(p+q)\right] \cdot \left[\frac{1}{2} B(p,q)\right]$$

$$\Rightarrow \quad B(p,q) = \frac{\Gamma(p)\,\Gamma(q)}{\Gamma(p+q)}$$