Lecture 2

[vector operators in orthogonal curvilinear coordinates]

Gradient $\vec{\nabla} f$

Divergence $\vec{\nabla} \cdot \vec{V}$

Curl $\vec{\nabla} \times \vec{V}$

Laplacian $\nabla^2 f$

Gradient: consider a function in 3D $\phi(x, y, z)$.

suppose you want to figure out how this function changes as you travel a distance s in some direction.

Let's start at a point (x_0, y_0, z_0), now go to a point (x, y, z) which is at a distance s in the direction of $\vec{u}$!

$$\Rightarrow \quad (x - x_0)\hat{i} + (y - y_0)\hat{j} + (z - z_0)\hat{k} = s(\vec{u})$$

$$\text{Let } \vec{u} = a\hat{i} + b\hat{j} + c\hat{k}$$

$$\Rightarrow \begin{bmatrix} x = x_0 + sa \\ y = y_0 + sb \\ z = z_0 + sc \end{bmatrix}$$

we can replace (x, y, z) in ϕ with $\Rightarrow \phi(x_0 + sa, \ldots)$

since x_0, a, y_0, b, z_0, c all are constant.

$\Rightarrow$ ϕ depends on the distance s

$$\frac{d\phi}{ds} = \frac{\partial \phi}{\partial x}\frac{\partial x}{\partial s} + \cdots$$

$$= \frac{\partial \phi}{\partial x} a + \frac{\partial \phi}{\partial y} b + \frac{\partial \phi}{\partial z} c$$

$$= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)(\phi) \cdot \vec{u}$$

$$\boxed{\frac{d\phi}{ds} = \vec{\nabla}\phi \cdot \vec{u}}$$

if $\vec{u}$ is a unit vector $\Rightarrow$ maximum change of ϕ would be in the direction of $\vec{\nabla}\phi$.

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now let us define a surface ϕ = constant

$$\Rightarrow \quad \frac{\partial \phi}{\partial s} = 0 = \vec{\nabla} \cdot \phi \cdot \vec{u}$$

where $\vec{u}$ is the tangent vector to that surface!

$\Rightarrow$ $\vec{\nabla}\phi$ is perpendicular to surface.

Another way to understand this is the following:

suppose, you chose ϕ = constant, and now choosing a vector in the direction suchthat it remains on the surface?

$$\Rightarrow \quad (x - x_0) = s\vec{u}$$

this is also on the plane!

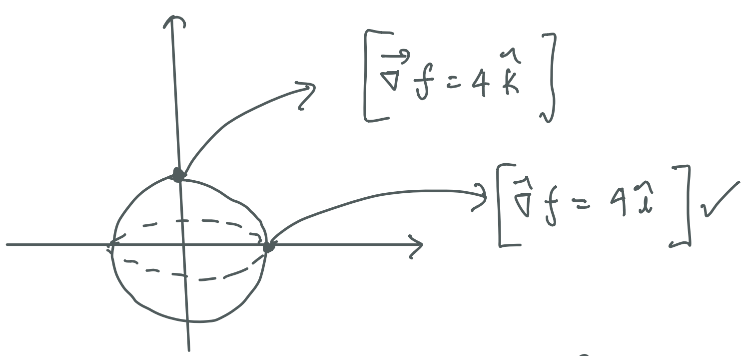
$\Rightarrow$ tangent vector!

(only possible direction left is tangential!) ✓

Example:

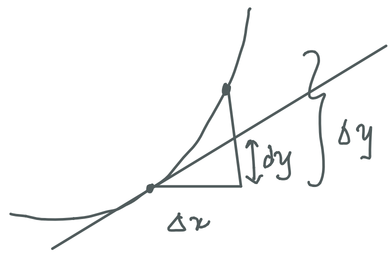
$$f = \underbrace{x^2+y^2+z^2-4}_{} = 0$$

$$\vec{\nabla} f = \underbrace{2x\hat{i} + 2y\hat{j} + 2z\hat{k}}_{}$$



what's the tangent vector?

What does $\frac{d\phi}{ds}$ mean?



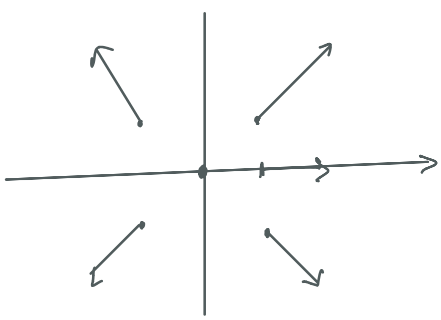
$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

$\boxed{dx = \Delta x} \to \Delta x \to 0 \equiv \boxed{dy = \Delta y}$

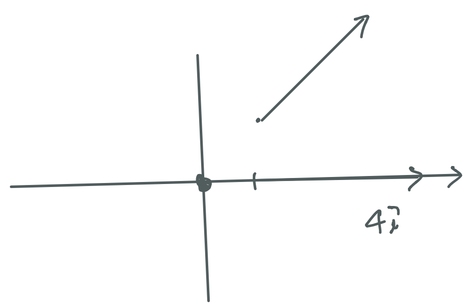
Let's talk about vector fields:

A) $\vec{V} = x\hat{i} + y\hat{j}$

$\left[\vec{\nabla} \cdot \vec{V} = 2\right]$



B) $\vec{V} = 4(x\hat{i} + y\hat{j})$



$\vec{\nabla} \cdot \vec{V} = 4$

so, it's easy to see that divergence tells us about how fast a vector field goes away from the source! Similarly, one can think about curl!

(HW) Try to visualize these in the context of Electric and magnetic field!

Ok, so we already studied how a function changes if we move a distance s in a particular direction. $\left[\frac{d\phi}{ds} = \vec{\nabla}\phi \cdot \vec{u}\right]$

However, $\vec{\nabla}$ was defined using Cartesian coordinate system.

Can we convert this in curvilinear coordinate system?

if we go in 'r' direction: $\Rightarrow$ $[ds = dr]$

$\frac{d\phi}{ds}$ is the component of $\vec{\nabla}\phi$ in that direction!

$\Rightarrow$ for $ds = dr$, $\vec{\nabla}\phi$ should be $\frac{\partial\phi}{\partial r}$?

for 'θ' direction; $ds = r\,d\theta$ $\Rightarrow$ $\vec{\nabla}\phi$ should be $\frac{\partial\phi}{r\,\partial\theta}$

for 'z' direction; $ds = dz$ $\Rightarrow$ $\vec{\nabla}\phi$ should be $\frac{\partial\phi}{\partial z}$

$$\Rightarrow \quad \vec{\nabla}\phi = \frac{\partial\phi}{\partial r}\hat{e}_r + \frac{\partial\phi}{r\,\partial\theta}\hat{e}_\theta + \frac{\partial\phi}{\partial z}\hat{e}_z \quad \text{(Great!)}$$

and we can even write general conditions.

Recall from the previous lecture!

$$ds^2 = h_1^2\,dx_1^2 + h_2^2\,dx_2^2 + h_3^2\,dx_3^2$$

$\Rightarrow$ $ds = h_1\,dx_1$ | $ds = h_2\,dx_2$ | , $ds = h_3\,dx_3$ | distance

$\hat{x}_1$ | $\hat{x}_2$ | $\hat{x}_3$ | direction

$\Rightarrow$ $\vec{\nabla}\phi$ in any curvilinear coordinate system!

$$\left[\vec{\nabla}\phi = \frac{1}{h_1}\frac{\partial\phi}{\partial x_1}\hat{x}_1 + \frac{1}{h_2}\frac{\partial\phi}{\partial x_2}\hat{x}_2 + \frac{1}{h_3}\frac{\partial\phi}{\partial x_3}\hat{x}_3\right]$$

$$\left[\vec{\nabla}\phi = \sum_{j=1}^{3} \frac{1}{h_i}\frac{\partial\phi}{\partial x_i}\hat{x}_i\right] \checkmark$$

How about Divergence, curl and Laplacian?

$\vec{\nabla}\cdot\vec{V}$? Let $\vec{V} = V_1\hat{e}_1 + V_2\hat{e}_2 + V_3\hat{e}_3$ (assuming an orthogonal coordinate system)

we can prove: $\vec{\nabla}\cdot\left[\frac{\hat{e}_3}{h_1h_2}\right] = 0 = \vec{\nabla}\cdot\left[\frac{\hat{e}_2}{h_1h_3}\right] = \vec{\nabla}\cdot\left(\frac{\hat{e}_1}{h_2h_3}\right)$

use $\phi = x_1$, x_2, x_3 three different cases

then $\vec{\nabla}\phi = \frac{\hat{e}_1}{h_1}$ $\frac{\hat{e}_2}{h_2}$ $\frac{\hat{e}_3}{h_3}$ $\checkmark$

then $\qquad \vec{\nabla} x_1 \otimes \vec{\nabla} x_3 = \dfrac{\hat{e}_1}{h_1} \otimes \dfrac{\hat{e}_2}{h_2} = \left[\dfrac{\hat{e}_3}{h_1 h_2}\right]$

now $\qquad \vec{\nabla} \cdot \left[\vec{\nabla} x_1 \otimes \vec{\nabla} x_3\right] = \vec{\nabla} x_3 \cdot \underbrace{\left[\vec{\nabla} \otimes \vec{\nabla} x_1\right]}_{=0} - \vec{\nabla} x_1 \cdot \underbrace{\left(\vec{\nabla} \otimes \vec{\nabla} x_3\right)}_{=0}$

$$\Rightarrow \left[\vec{\nabla} \cdot \left[\frac{\hat{e}_3}{h_1 h_2}\right] = 0\right] \checkmark$$

$$\Rightarrow \quad V = \frac{\hat{e}_1}{h_2 h_3}\left(h_2 h_3 V_1\right) + \frac{\hat{e}_2}{h_1 h_3} h_1 h_3 V_2 + \frac{\hat{e}_3}{h_1 h_2}\left(h_1 h_2 V_3\right)$$

$$\because \vec{\nabla} \cdot (\phi \vec{v}) = \vec{v} \cdot (\vec{\nabla} \phi) + \phi \vec{\nabla} \cdot \vec{v}$$

$$= \vec{\nabla} \cdot \left[h_2 h_3 V_1 \frac{\hat{e}_1}{h_2 h_3}\right] = \underbrace{\frac{\hat{e}_1}{h_2 h_3} \cdot \vec{\nabla}(h_2 h_3 V_1)} + h_2 h_3 V_1 \underbrace{\vec{\nabla} \cdot \frac{\hat{e}_1}{h_2 h_3}}_{0, \text{ we proved!}}$$

$$\Rightarrow \frac{1}{h_1 h_2 h_3}\frac{\partial}{\partial x_1}\left(h_2 h_3 V_1\right) + \frac{1}{h_1 h_2 h_3}\frac{\partial}{\partial x_2}\left(h_1 h_3 V_2\right) + \frac{1}{h_1 h_2 h_3}\frac{\partial}{\partial x_3}\left(h_1 h_2 V_3\right) \checkmark$$

Ex! Let's do a quick exercise:

find $\vec{\nabla} \cdot \vec{v}$ in cylindrical coordinate system!

Laplacian: $\vec{\nabla}\cdot(\vec{\nabla}\phi) = \nabla^2\phi$

$\because \vec{\nabla}\phi = \frac{1}{h_i}\frac{\partial\phi}{\partial x_i}$

then $\vec{\nabla}\cdot\vec{\nabla}\phi = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial x_1}\left[\frac{h_2 h_3}{h_1}\frac{\partial\phi}{\partial x_1}\right]\right]$

Ex: find for cylindrical coordinate system.

Curl:

$$\vec{\nabla}\times V = \frac{1}{h_1 h_2 h_3}\begin{vmatrix} h_1 e_1 & h_2 e_2 & h_3 e_3 \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ h_1 v_1 & h_2 v_2 & h_3 v_3 \end{vmatrix}$$

Sol: $\because \vec{\nabla} x_1 = \frac{\hat{e}_1}{h_1}$

$\vec{\nabla}\times\frac{\hat{e}_1}{h_1} = 0$

because

$\vec{\nabla}\otimes(\vec{\nabla}\phi) = 0$

$\vec{V} = \frac{\hat{e}_1}{h_1}(h_1 v_1) + \frac{\hat{e}_2}{h_2}(h_2 v_2) + \frac{\hat{e}_3}{h_3}(h_3 v_3)$

$\vec{\nabla}\times(\phi\vec{V}) = \phi(\vec{\nabla}\times\vec{V}) - \vec{V}\times(\vec{\nabla}\phi)$

$$\vec{\nabla}\times\vec{V} = -\frac{\hat{e}_1}{h_1}\times\vec{\nabla}(h_1 v_1) - \frac{\hat{e}_2}{h_2}\times\vec{\nabla}(h_2 v_2) - \frac{\hat{e}_3}{h_3}\times\vec{\nabla}(h_3 v_3)$$

$$-\frac{\hat{e}_3}{h_1}\left[\frac{1}{h_2}\frac{\partial}{\partial x_2}(h_1 v_1)\right]+\frac{\hat{e}_2}{h_1}\left[\frac{1}{h_3}\frac{\partial}{\partial x_3}(h_1 v_1)\right]$$

$\Rightarrow$ can check the other terms!
