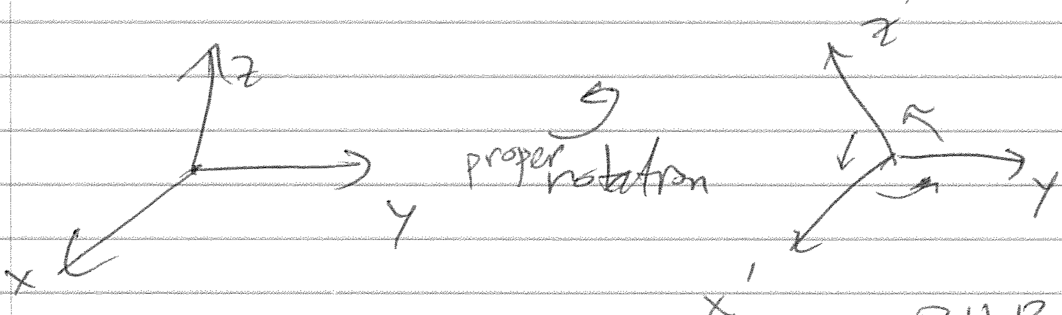
Lecture 6

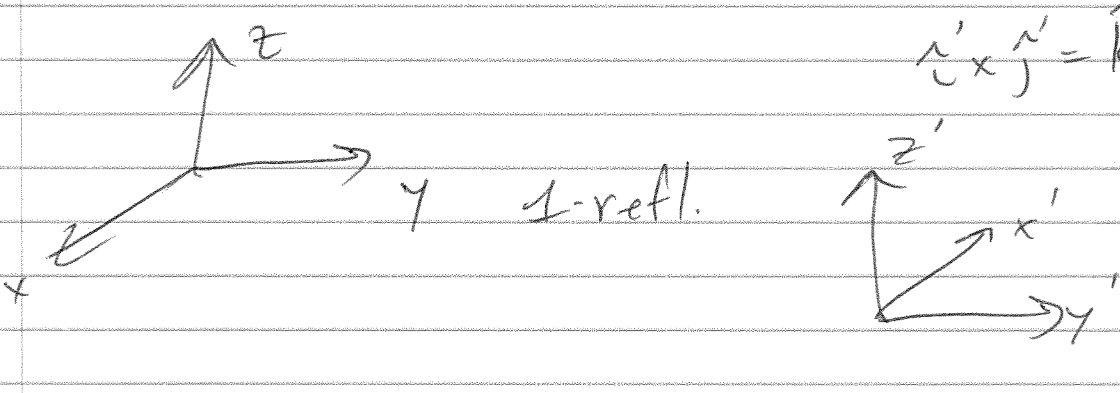
Pseudovectors & pseudotensors

An orthogonal transform preserves length & can always be decomposed into a proper rotation or 1, 2, or 3 reflections



RHR is OK

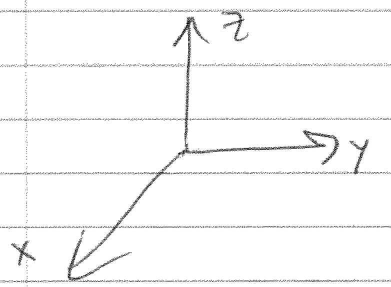
$\hat{i}' \times \hat{j}' = \hat{k}'$



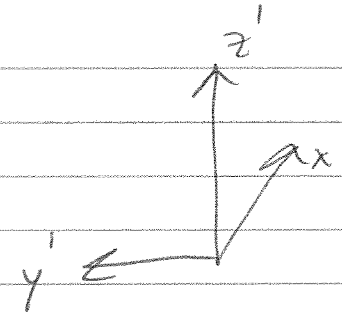
RHR violated

$\hat{i}' \times \hat{j}' = -\hat{k}$

improper

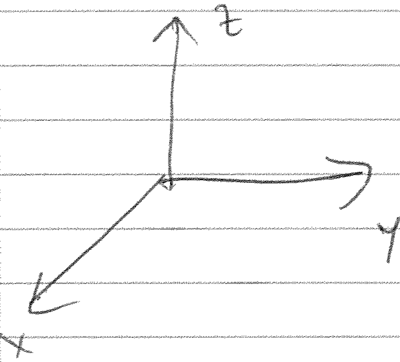


2-refl.

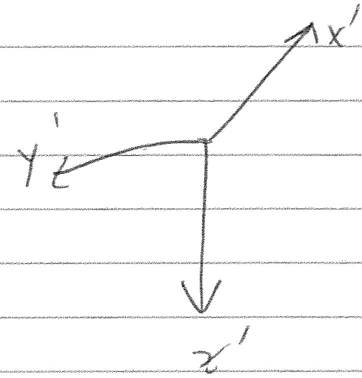


$\hat{i}' \times \hat{j}' = +\hat{k}'$

RHR OK proper.



3-refl.



$\hat{i}' \times \hat{j}' = -\hat{k}'$

RHR violated

improper.

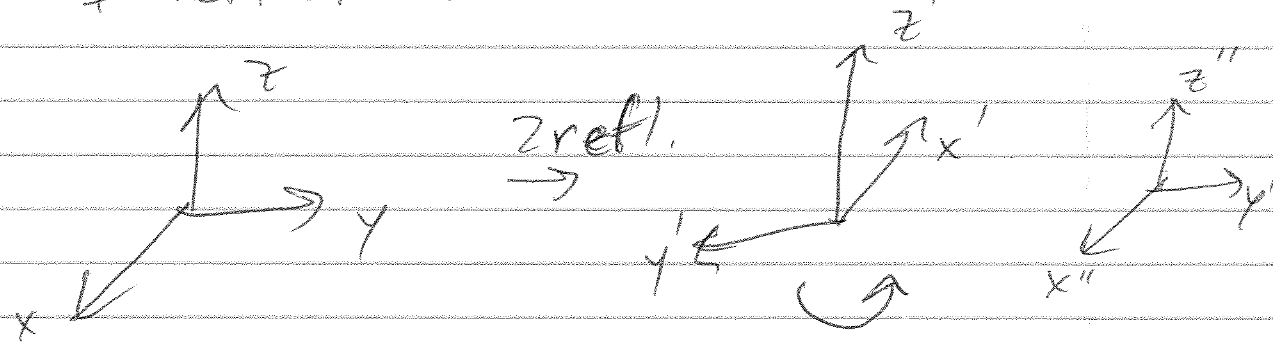
A reflection of 0 or 2 axes is called proper & maintains RHR.

A reflection of 1 or 3 (inversion) is called improper & violates RHR.

- If $\overleftrightarrow{A}$ is an orthogonal transform, then
  det $|\overleftrightarrow{A}| = 1$ proper &
  det $|\overleftrightarrow{A}| = -1$ improper

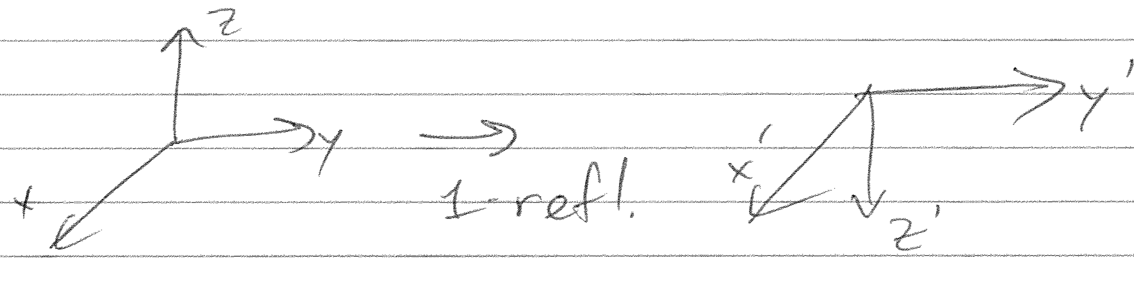
- Any proper transform can always be reduced to a pure rotation

- Any improper transform is always a combination of proper rotation & 1 reflection.



Example

- If $\vec{U}$ & $\vec{V}$ are proper vectors, then the cross-product is improper.
- That is, $\vec{W} = \vec{U} \times \vec{V}$ is not a proper rank-one tensor & is instead called pseudo-rank-one-tensor or pseudovector.
- Suppose we perform the reflection



$$\vec{U} = [U_1, U_2, U_3] \rightarrow \vec{U}' = [U_1, U_2, -U_3]$$

$$\vec{V} = [V_1, V_2, V_3] \rightarrow \vec{V}' = [V_1, V_2, -V_3]$$

But

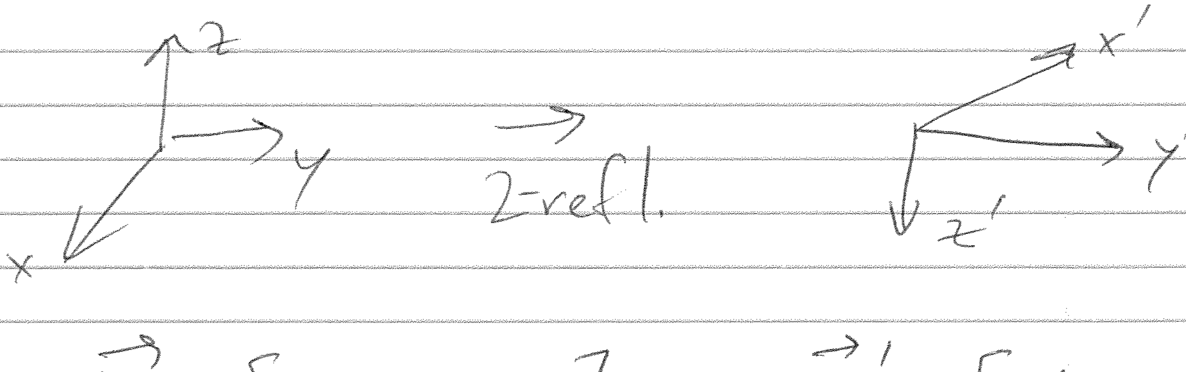
$$\vec{W} = [\overbrace{U_2V_3 - U_3V_2}^{W_1}, -(\overbrace{U_1V_3 - U_3V_1}^{W_2}), \underbrace{U_1V_2 - U_2V_1}_{W_3}]$$

& $\vec{W}' = \vec{U}' \times \vec{V}'$

$$= [-W_1, -W_2, W_3]$$

So then $\vec{W} = \vec{U} \times \vec{V}$ transforms differently under 1 refl. than $\vec{U}$ or $\vec{V}$.

If there are 2 reflections



$$\vec{U} = [U_1, U_2, U_3] \rightarrow \vec{U}' = [-U_1, U_2, -U_3]$$

$$\vec{V} = [V_1, V_2, V_3] \rightarrow \vec{V}' = [-V_1, V_2, -V_3]$$

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then

$$\vec{W} = [\underbrace{U_2V_3 - U_3V_2}_{= W_1},\ \underbrace{-(U_1V_3 - U_3V_1)}_{= W_2},\ \underbrace{U_1V_2 - U_2V_1}_{= W_3}]$$

$$\rightarrow \quad \vec{W}' = [-W_1,\ W_2,\ -W_3]$$

& so it transforms properly again.

We can alternatively describe this as

$$U_\beta' = a_{\beta m}\, U_m \qquad \text{proper}$$

$$V_\gamma' = a_{\gamma p}\, V_p \qquad \text{proper}$$

So then

$$(\vec{U}' \times \vec{V}')_\alpha = \epsilon'_{\alpha\beta\gamma}\, U_\beta' V_\gamma'$$

How does ϵ_{ijk} transform?

Recall that $\epsilon_{\alpha\beta\gamma} \det|\overleftrightarrow{A}|$

$$= a_{\alpha i}\, a_{\beta j}\, a_{\gamma k}\, \epsilon_{ijk}$$

If $\overleftrightarrow{A}$ is proper, then no sign, but if $\overleftrightarrow{A}$ is improper then we get a sign

So then

$$\epsilon'_{\alpha\beta\gamma} = \det|\overleftrightarrow{A}|\, a_{\alpha i}\, a_{\beta j}\, a_{\gamma k}\, \epsilon_{ijk}$$

So ϵ_{ijk} is a rank-three pseudo tensor.

So we found that

$$(\vec{U} \times \vec{V})_\alpha = \epsilon'_{\alpha\beta\gamma} U'_\beta V'_\gamma$$

$$= \left[\det|\overleftrightarrow{A}|\, a_{\alpha i}\, a_{\beta j}\, a_{\gamma k}\, \epsilon_{ijk}\right] \cdot \left[a_{\beta m} U_m\right]\left[a_{\gamma p} V_p\right]$$

$$= \det|\overleftrightarrow{A}|\, a_{\alpha i}\, \delta_{jm}\, \delta_{kp}\, \epsilon_{ijk}\, U_m V_p$$

$$= \det|\overleftrightarrow{A}|\, a_{\alpha i}\, \epsilon_{ijk}\, U_j V_k$$

$$= \det|\overleftrightarrow{A}|\, a_{\alpha i} \left[\vec{U} \times \vec{V}\right]_i$$

$\Rightarrow$

$$W'_\alpha = \det|\overleftrightarrow{A}|\, a_{\alpha i}\, W_i$$

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Terminology

- $\vec{U}$ & $\vec{V}$ are called polar or true or proper vectors if they transform properly.

- $\vec{W} = \vec{U} \times \vec{V}$ is called a pseudo or axial vector & transforms improperly.

Example

Let $\vec{A}, \vec{B}, \vec{C}, \vec{D}$ be polar/true/proper.

then $\vec{V} = \vec{A} \times \vec{B}$ &

$U = \vec{C} \times \vec{D}$ improper

But $\vec{W} = \vec{V} \times \vec{U}$

$$= (\vec{A}\times\vec{B})\times(\vec{C}\times\vec{D})$$

$$= \underbrace{\left[(\vec{A}\times\vec{B})\cdot\vec{D}\right]}_{\text{scalar}}\vec{C} - \underbrace{\left[(\vec{A}\times\vec{B})\cdot\vec{C}\right]}_{\text{scalar}}\vec{D}$$

= prop. − prop. = prop.

Examples

proper / polar / true	improper / pseudo
$\vec{r}$ position vector	$\vec{\theta}$ ∡ position
$\vec{\nabla}$ gradient vector	$\vec{L} = I\vec{\omega}$ ∡ momentum
$\vec{p} = m\vec{v}$ momentum vect.	$\vec{\omega} = \frac{d\vec{\theta}}{dt}$ ∡ velocity
$\vec{v}$ velocity	$\vec{\alpha} = \frac{d\vec{\omega}}{dt}$ ∡ acc.
$\vec{a}$ acceleration	
$\vec{F} = m\vec{a}$	$\vec{\tau} = I\vec{\alpha}$ torque
$\vec{E} = -\vec{\nabla}\Phi$ electric field	$\vec{B} = \vec{\nabla}\times\vec{A}$ mag. field ($\vec{A}$: vector potential)

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$$\vec{F} = q\vec{E}$$

$$\vec{F} = q\vec{v} \times \vec{B}$$

$$\frac{\partial \vec{B}}{\partial t} = -\vec{\nabla} \times \vec{E}$$

$$\vec{\tau} = \vec{r} \times \vec{F}$$

$$\vec{L} = m\,\vec{r} \times \vec{v}$$

$$= m\,\vec{r} \times (\vec{\omega} \times \vec{r})$$