Lecture 5

Products of Isotropic tensors

- Recall that the direct product of a tensor w/ rank n & one w/ rank m produces a tensor of rank n+m.
- Contraction reduces rank by two.

Useful formula

$$\epsilon_{ijk}\,\epsilon_{imn} = \delta_{jm}\,\delta_{kn} - \delta_{jn}\,\delta_{km}$$

$\epsilon_{ijk}\,\epsilon_{lmn}$ is a rank 6 tensor & setting $i=l$ makes a rank 4 tensor.

Show proof in Mathematica
idea is to take difference of LHS & RHS & show that it is zero.

can prove that

$\epsilon_{ijk}\,\epsilon_{ijn} = 2\,\delta_{kn}$

& then $\epsilon_{ijk}\,\epsilon_{ijk} = 6$

Dot product is given by $\vec{A}\cdot\vec{B} = A_i B_j \delta_{ij}$

We can express the cross product using the Levi-Civita symbol.

$(B\times C)_i = \epsilon_{ijk} B_j C_k$

To prove this, recall relation to determinant & set

$\vec{A} = \vec{B}\times\vec{C}$

$$\begin{array}{ccc} + & - & + \\ \end{array}$$

$$\begin{pmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{pmatrix}$$

$$A_x = + \left[B_y C_z - B_z C_y \right]$$

$$A_y = - \left[B_x C_z - B_z C_x \right]$$

$$A_z = + \left[B_x C_y - B_y C_x \right]$$

Then $A_i = \epsilon_{ijk} B_j C_k$

$$A_1 = \epsilon_{123} B_2 C_3 + \epsilon_{132} B_3 C_2$$

Any repeated indices for ϵ_{ijk} vanish so no ones there.

$$= B_2 C_3 - B_3 C_2 = B_y C_z - B_z C_y$$

Similarly,

$$A_2 = \epsilon_{231} B_3 C_1 + \epsilon_{213} B_1 C_3$$

$$= B_3 C_1 - B_1 C_3 = B_z C_x - B_x C_z$$

$$A_3 = \epsilon_{312} B_1 C_2 + \epsilon_{321} B_2 C_1$$

$$= B_1 C_2 - B_2 C_1 = B_x C_y - B_y C_x$$

~~can write~~ $\cancel{[\hat{x}, \hat{y}, \hat{z}]}$ ~~as~~ $\cancel{[\hat{e}_1, \hat{e}_2, \hat{e}_3]}$

~~& then~~ $\cancel{\vec{A}}$

Now prove that

$$\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})$$

Let $\vec{D} = \vec{B} \times \vec{C} \Rightarrow D_i = \epsilon_{ijk} B_j C_k$

& then

$\vec{F} = \vec{A} \times \vec{D} \Rightarrow F_l = \epsilon_{lmn} A_m D_n$

$$\Rightarrow F_l = \epsilon_{lmn} A_m D_n$$

$$= \epsilon_{lmn} \epsilon_{njk} A_m B_j C_k$$

$$= \epsilon_{nlm} \epsilon_{njk} A_m B_j C_k$$

$$= \left[\delta_{lj} \delta_{mk} - \delta_{lk} \delta_{mj} \right] A_m B_j C_k$$

$$= \delta_{lj} \delta_{mk} A_m B_j C_k - \delta_{lk} \delta_{mj} A_m B_j C_k$$

$$= B_l A_k C_k - C_l A_m B_m$$

$$\Rightarrow \vec{F} = \vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B})$$

Can use this to find alternate representation of curl curl. (captures infinitesimal rotation of a vector field)

Recall that curl is given by

$$(\underline{\nabla} \times \underline{V})_i = \epsilon_{ijk} \frac{\partial}{\partial x_j} V_k$$

Then curl of curl is

$$\nabla \times (\nabla \times \underline{V})$$

Set $\vec{D} \equiv \nabla \times \underline{V}$

& $\vec{F} = \nabla \times \vec{D}$

$$F_\ell = \epsilon_{\ell mn} \frac{\partial}{\partial x_m} D_n$$

$$D_i = \epsilon_{ijk} \frac{\partial}{\partial x_j} V_k$$

$$\Rightarrow F_\ell = \epsilon_{\ell mn} \epsilon_{njk} \frac{\partial}{\partial x_m} \frac{\partial}{\partial x_j} V_k$$

$$= \epsilon_{n\ell m} \epsilon_{njk} \frac{\partial}{\partial x_m} \frac{\partial}{\partial x_j} V_k$$

$$= \left[\delta_{\ell j} \delta_{mk} - \delta_{\ell k} \delta_{mj} \right] \frac{\partial}{\partial x_m} \frac{\partial}{\partial x_j} V_k$$

$$= \delta_{\ell j} \delta_{mk} \frac{\partial}{\partial x_m} \frac{\partial}{\partial x_j} V_k -$$

$$\delta_{\ell k} \delta_{mj} \frac{\partial}{\partial x_m} \frac{\partial}{\partial x_j} V_k$$

$$= \frac{\partial}{\partial x_m} \frac{\partial}{\partial x_\ell} V_m - \frac{\partial}{\partial x_m} \frac{\partial}{\partial x_m} V_\ell$$

$$= \frac{\partial}{\partial x_\ell} \left[\frac{\partial}{\partial x_m} V_m \right] - \frac{\partial}{\partial x_m} \frac{\partial}{\partial x_m} V_\ell$$

$$\Rightarrow \vec{F} = \underline{\nabla}\underbrace{(\underline{\nabla}\cdot\underline{V})}_{\text{divergence}} - \underbrace{\nabla^2}_{\text{Laplacian}}\underline{V}$$

$$\nabla^2 = \sum_m \frac{\partial^2}{\partial x_m^2}$$

Dual tensors

Let T_{ij} be an antisymmetric rank-two tensor so that

$T_{ij} = -T_{ji}$

Writing out components of T_{ij} as a matrix gives

$$T = \begin{bmatrix} 0 & T_{12} & -T_{31} \\ -T_{12} & 0 & T_{23} \\ T_{31} & -T_{23} & 0 \end{bmatrix}$$

Thus, there are only 3 independent components.

Can then define the vector

$V_i = \frac{1}{2} \epsilon_{ijk} T_{jk}$

$\Rightarrow \quad V_1 = T_{23}, \quad V_2 = T_{31}, \quad V_3 = T_{12}$

Since ϵ_{ijk} is rank three
& T_{lm} is rank two
$\epsilon_{ijk} T_{lm}$ is rank five &
two contractions give rank one
tensor (vector)

V_k is called dual of T_{ij}

Conversely, we can ~~start~~ start w/
a vector V_k & define T_{ij} as

$T_{ij} = \epsilon_{ijk} V_k$

guaranteed to be an antisymmetric
rank two tensor.