Lecture 4

Example of calculating inertia tensor for two point masses

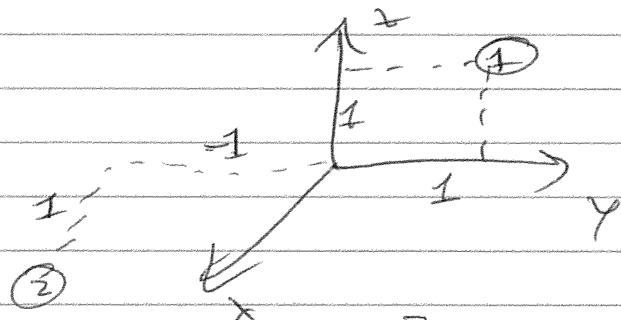
Suppose mass $m_1 = 1$ is located

@ $\vec{r}_1 = [0, 1, 1] = [x^1, y^1, z^1]$
$= [x_1^{(1)}, x_2^{(1)}, x_3^{(1)}]$

Suppose mass $m_2 = 2$ is located

@ $\vec{r}_2 = [1, -1, 0] = [x^2, y^2, z^2]$
$= [x_1^{(2)}, x_2^{(2)}, x_3^{(2)}]$

Find $\overleftrightarrow{I}$ in this frame



$$I_{ij}^{(1)} = m_1 [r_1^2 \delta_{ij} - x_i^{(1)} x_j^{(1)}]$$

$$= 1 \cdot 2 \cdot \delta_{ij} - x_i^{(1)} x_j^{(1)}$$

(2)

$$= 2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} xx & xy & xz \\ xy & yy & yz \\ xz & yz & zz \end{bmatrix}$$

$$= 2\begin{bmatrix} & " & \end{bmatrix} - \begin{bmatrix} 0\cdot 0 & 0\cdot 1 & 0\cdot 1 \\ 0 & 1\cdot 1 & 1\cdot 1 \\ 0 & 1\cdot 1 & 1\cdot 1 \end{bmatrix}$$

$$= 2\begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

Mass 2

$$I_{ij}^{(2)} = 2\left[2 \cdot \delta_{ij} - x_i^{(2)} x_j^{(2)}\right]$$

$$= 4\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - 2\begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} + \begin{bmatrix} -2 & 2 & 0 \\ 2 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$I_{ij} = I_{ij}^{(1)} + I_{ij}^{(2)} = \begin{bmatrix} 4 & 2 & 0 \\ 2 & 3 & -1 \\ 0 & -1 & 5 \end{bmatrix}$$

Note that a superposition principle applies, so that $\overleftrightarrow{I} = \overleftrightarrow{I}^{(1)} + \overleftrightarrow{I}^{(2)} + \overleftrightarrow{I}^{(3)} + \dots$

can compute eigenvectors & eigenvalues using Mathematica.

We get

$\lambda_1 = 6$,

$\lambda_2 = 3 + \sqrt{3}$, &

$\lambda_3 = 3 - \sqrt{3}$

So there exists a coordinate system (x', y', z') in which $\overleftrightarrow{I} \to \overleftrightarrow{I}'$ is diagonal

$$\overleftrightarrow{I}' = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 3+\sqrt{3} & 0 \\ 0 & 0 & 3-\sqrt{3} \end{bmatrix} = \begin{bmatrix} I'_{x'x'} & 0 & 0 \\ 0 & I'_{y'y'} & 0 \\ 0 & 0 & I'_{z'z'} \end{bmatrix}$$

Principal moments of inertia are then

$$I'_{ij} = I'_{ij}\,\delta_{ij}$$

Normalized eigenvectors are

$$\hat{i}' = [-1, -1, 1]/\sqrt{3}$$

$$\hat{j}' = \frac{[-1+\sqrt{3},\ 2-\sqrt{3},\ 1]}{\sqrt{12-6\sqrt{3}}}$$

$$\hat{k}' = \frac{[-1-\sqrt{3},\ 2+\sqrt{3},\ 1]}{\sqrt{12+6\sqrt{3}}}$$

so that ~~rotation~~ matrix is

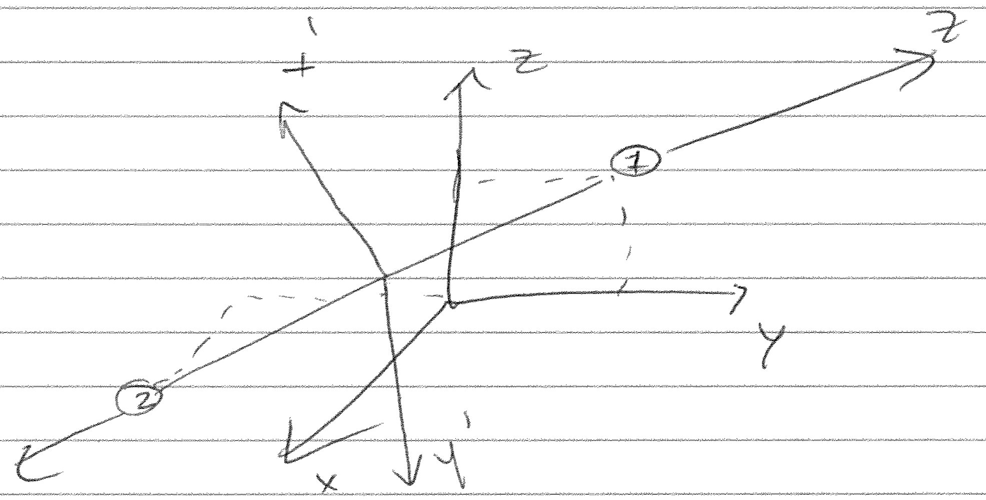
$$\begin{bmatrix} -1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3} \\ \dfrac{-1+\sqrt{3}}{\sqrt{12-6\sqrt{3}}} & \dfrac{2-\sqrt{3}}{\sqrt{12-6\sqrt{3}}} & \dfrac{1}{\sqrt{12-6\sqrt{3}}} \\ \dfrac{-1-\sqrt{3}}{\sqrt{12+6\sqrt{3}}} & \dfrac{2+\sqrt{3}}{\sqrt{12+6\sqrt{3}}} & \dfrac{1}{\sqrt{12+6\sqrt{3}}} \end{bmatrix} = \{a_{ij}\}$$

+ then

$$I'_{ij} = a_{ik} a_{jl} I_{kl}$$

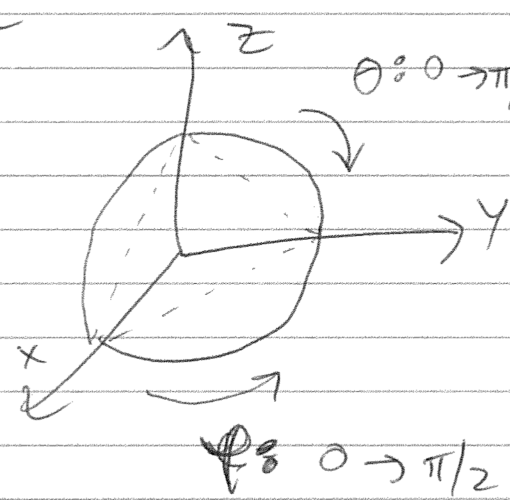
(Einstein convention)

then picture 3



Another example

Moment of Inertia tensor for an octent of a sphere



Let mass density be $\rho(\vec{r})$ inside

& $\rho(\vec{r})$ outside

Task: Find $\overleftrightarrow{I}$ & then $\overleftrightarrow{I}'$ &

$x', y', z' \leftrightarrow x, y, z$

By symmetry, it should be the case that one principal axis is

$\hat{k}' = [1, 1, 1]/\sqrt{3}$

& then any 2 orthogonal vectors in the plane $x+y+z=0$ b/c we should have $\hat{i}' \cdot \hat{k}' = 0$ & $\hat{j}' \cdot \hat{k}' = 0$

e.g. $\hat{i}' = [1, -1, 0]/\sqrt{2}$

$\hat{j}' = [1, 1, -2]/\sqrt{4}$

Now calculate

$$I_{xx} = \int_V dr^3 \, \rho(\vec{r}) \, (r^2 - x^2)$$

use spherical coordinates

$\Rightarrow x = r \sin\theta \cos\varphi$

$\Rightarrow x^2 = r^2 \sin^2\theta \cos^2\varphi$

& $dr^3 = dV = r^2 dr \sin\theta \, d\varphi \, d\theta$

$$\Rightarrow I_{xx} = \int_{r=0}^{1} \int_{\theta=0}^{\pi/2} \int_{\varphi=0}^{\pi/2} r^4 \left[1 - \sin^2\theta \cos^2\varphi\right] \sin\theta \, dr \, d\theta \, d\varphi$$

$$= \pi/15$$

$$I_{xy} = - \int_V dV \, \rho(\vec{r}) \, xy$$

$$= - \int_{r=0}^{1} \int_{\theta=0}^{\pi/2} \int_{\varphi=0}^{\pi/2} r^4 \sin^3\theta \sin\varphi \cdot \cos\varphi \, dr \, d\theta \, d\varphi$$

$$= -1/15$$

$x = r \sin\theta \cos\varphi$

$y = r \sin\theta \sin\varphi$

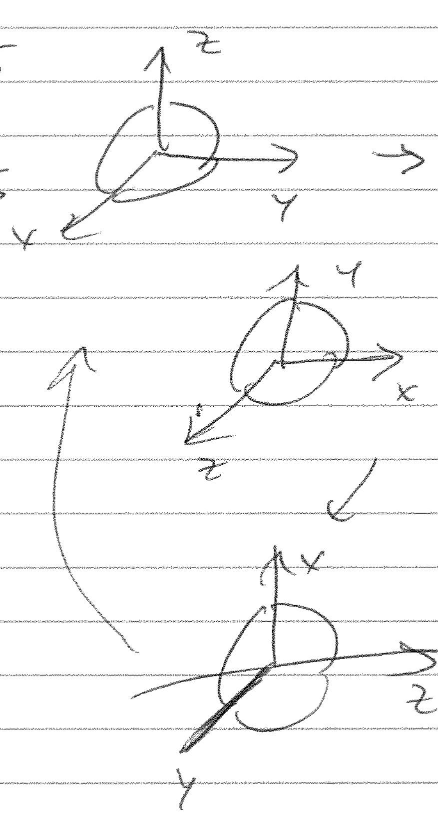
$dV = r^2 \sin\theta \, dr \, d\theta \, d\varphi$

Due to symmetry considerations, this is all we need to calculate

$\overleftrightarrow{I}$ is invariant under $x \to y \to z$ (and $z \to x$)

$\Rightarrow I_{xx} = I_{yy} = I_{zz} = \pi/15$

$I_{xy} = I_{xz} = I_{yz} = -1/15$



$$\Rightarrow \overleftrightarrow{I} = \frac{1}{15}\begin{bmatrix} \pi & -1 & -1 \\ -1 & \pi & -1 \\ -1 & -1 & \pi \end{bmatrix}$$

Eigenvalues are

$I_{x'x'} = \dfrac{\pi+1}{15}$, $I_{y'y'} = \dfrac{\pi+1}{15}$,

$I_{z'z'} = \dfrac{\pi-2}{15}$ so that

$$\overleftrightarrow{I}' = \frac{1}{15}\begin{bmatrix} \pi+1 & 0 & 0 \\ 0 & \pi+1 & 0 \\ 0 & 0 & \pi-2 \end{bmatrix}$$

$\not\!\!+$ $\hat{i}' = [-1, 0, 1]/\sqrt{2}$

$\hat{j}' = [-1, 1, 0]/\sqrt{2}$

$\hat{k}' = [1, 1, 1]/\sqrt{3}$

so that

$$\overleftrightarrow{A} = \begin{bmatrix} -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \end{bmatrix} = \{a_{ij}\}$$

$\not\!\!+$

$$I'_{ij} = a_{ik} a_{jl} I_{kl}$$

Kronecker Delta & Levi-Civita

Kronecker delta is used in tensor contractions

$$\delta_{ij} = \begin{cases} 1 & i=j \\ 0 & i\neq j \end{cases}$$

$$\{\delta_{ij}\} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ 2nd rank tensor

In cross products & determinants, the Levi-Civita symbol appears

$$\underbrace{\epsilon_{ijk}}_{\text{rank three tensor}} \equiv \begin{cases} 1, & ijk = 123, 312, 231, 123 \\ -1, & ijk = 213, 321, 132, 213 \\ 0, & i=j \text{ or } j=k \text{ or } i=k \end{cases}$$

δ_{ij} & ϵ_{ijk} are isotropic tensors

$$\delta'_{ij} = \delta_{ij}$$

$$\epsilon'_{ijk} = \epsilon_{ijk}$$ under coordinate transformations

Proof:

$$\delta'_{mn} = a_{mi}\, a_{nj}\, \delta_{ij} \qquad \text{(Einstein)}$$

$$= a_{mi}\, a_{ni}$$

$$= \delta_{mn}$$ by orthonormality of rows + columns of $\overleftrightarrow{A}$

Thus δ_{ij} is an isotropic tensor of rank two.

Determinants

3x3 matrix determinants can be written using ϵ_{ijk}

Let $\overleftrightarrow{A} = \{a_{ij}\}$ be a 3x3 matrix

then
$$\det \begin{array}{c} \begin{array}{ccc} + & - & + \end{array} \\ \left| \begin{array}{ccc} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{array} \right. \end{array}$$

$$= a_{11}\left[a_{22}a_{33} - a_{32}a_{23}\right]$$

$$-a_{12}\left\{a_{21}a_{33} - a_{31}a_{23}\right]$$

$$+a_{13}\left[a_{21}a_{32} - a_{31}a_{22}\right]$$

$$= a_{11}a_{22}a_{33} - a_{11}a_{32}a_{23} - a_{12}a_{21}a_{33}$$

$$+a_{12}a_{31}a_{23} + a_{13}a_{21}a_{32}$$

$$-a_{13}a_{31}a_{22}$$

Now consider

$a_{1i}\, a_{2j}\, a_{3k}\, \epsilon_{ijk}$ (Einstein)

27 terms but only six survive (where $i \neq j \neq k$)

there are 6 of them

even perm's (+)		odd perm's (−)	
123	$a_{11}\, a_{22}\, a_{33}$	213	$a_{12}\, a_{21}\, a_{33}$
231	$a_{12}\, a_{23}\, a_{31}$	321	$a_{13}\, a_{22}\, a_{31}$
312	$a_{13}\, a_{21}\, a_{32}$	132	$a_{11}\, a_{23}\, a_{32}$

then

$$\det|\overleftrightarrow{A}| = \sum_{ijk} a_{1i}\, a_{2j}\, a_{3k}\, \epsilon_{ijk}$$

$$= a_{1i}\, a_{2j}\, a_{3k}\, \epsilon_{ijk}$$

(Einstein)

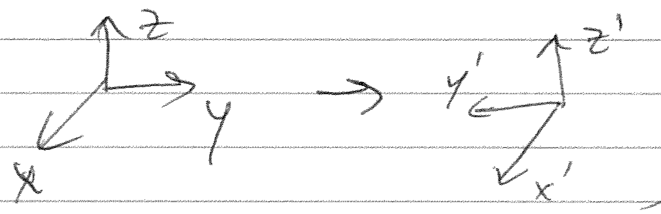
Let $\overleftrightarrow{A}$ now denote a proper rotation matrix (proper means no reflection)

then

$$\det|\overleftrightarrow{A}| = 1 \qquad \text{(definition)}$$

Physically $\left|\det|\overleftrightarrow{A}|\right| = 1$ means that rotation matrix preserves orientation & $\det|\overleftrightarrow{A}| = -1$ means that the transform is "proper"

[$\det|\overleftrightarrow{A}| = -1$ is improper, e.g.,



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Then under rotations, ϵ_{ijk} transforms as

$$a_{\alpha i}\, a_{\beta j}\, a_{\gamma k}\, \epsilon_{ijk}$$

~~Since $\det|\overleftrightarrow{A}|$~~

It can be shown that this collapses to

$$\epsilon_{\alpha\beta\gamma} \det|\overleftrightarrow{A}|$$

Since $\det|\overleftrightarrow{A}| = 1$, conclude that $\epsilon_{\alpha\beta\gamma}$ & so isotropic