Lecture 2

Cartesian vector (Cartesian tensor of rank one)

$\underline{V}$ consists of a set of 3 numbers or components in every coordinate system.

- If V_x, V_y, V_z are components in one system & V_x', V_y', V_z' are components in a rotated system, then they are related by a rotation matrix:

$$\begin{pmatrix} V_x' \\ V_y' \\ V_z' \end{pmatrix} = A \begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix} \qquad V' = AV$$

Alternatively, $V = A^T V'$

We can use the following notation as well to describe the relation between V' + V

$$V_x, V_y, V_z \to V_1, V_2, V_3$$

$$V_x', V_y', V_z' \to V_1', V_2', V_3'$$

$$A \to \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$$\Rightarrow \quad V_i' = \sum_{j=1}^{3} a_{ij} V_j \qquad i = 1, 2, 3$$

Alternatively, by using that $V = A^T V'$, we have that

$$V_i = \sum_{j=1}^{3} a_{ji} V_j'$$

Cartesian Tensors

- A tensor of rank zero (scalar) has one component & is unchanged by rotation of axes

- Examples: length of a vector or dot product of two vectors

- A tensor of rank two has 9 components in every coordinate system.
- Let T_{ij} denote components in one coordinate system, & let T'_{kl} denote components in another coordinate system

They are related by

$$T'_{k\ell} = \sum_{i=1}^{3} \sum_{j=1}^{3} a_{ki}\, a_{\ell j}\, T_{ij}$$

$$k, \ell = 1, 2, 3$$

In matrix notation, this would be

$$T' = A T A^T$$

Another example of a rank-two tensor is given by direct product (outer product or tensor product) of two vectors

Given vectors $\underline{u}$ + $\underline{v}$ w/ representations u_1, u_2, u_3 + v_1, v_2, v_3 in some coordinate system.

We can then form the following array:

$$\begin{array}{ccc} U_1 V_1 & U_1 V_2 & U_1 V_3 \\ U_2 V_1 & U_2 V_2 & U_2 V_3 \\ U_3 V_1 & U_3 V_2 & U_3 V_3 \end{array}$$

In linear algebra you would write this as

$$\begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix} \begin{bmatrix} V_1 & V_2 & V_3 \end{bmatrix}$$

these are the components of a rank-two tensor denoted by $\underline{U}\,\underline{V}$. (different from dot product or cross product)

Why is this a tensor?

(6)

Components of $\underline{U}$ & $\underline{V}$ in a rotated coordinate system are given by

$$U_k' = \sum_{i=1}^{3} a_{ki} U_i \qquad V_\ell' = \sum_{j=1}^{3} a_{\ell j} V_j$$

Components of the 2nd rank tensor $\underline{UV}$ in rotated coordinate system are then

$$U_k' V_\ell' = \left(\sum_{i=1}^{3} a_{ki} U_i\right)\left(\sum_{j=1}^{3} a_{\ell j} V_j\right)$$

$$= \sum_{i,j=1}^{3} a_{ki} a_{\ell j} U_i V_j$$

but this is the same as what we said previously should hold for a rank-two tensor

$$T_{ij} = U_i V_j \quad \& \quad T_{k\ell}' = U_k' V_\ell'$$

These equations generalize to higher rank tensors.

- For example, for a rank-four tensor T_{ijkl}, we get from this representation in a given coordinate system to another representation in a different coordinate system via

$$T'_{\alpha\beta\gamma\delta} = \sum_{ijkl} a_{\alpha i}\, a_{\beta j}\, a_{\gamma k}\, a_{\delta l}\, T_{ijkl}$$

10.3 Tensor notation & operations

Einstein Summation convention

Since so many summation symbols appear when manipulating tensors, it can simplify the notation to eliminate them.

- Convention is that summation is implicit or understood whenever there are repeated indices

Examples:

1) a_{ii} or a_{jj} or $a_{\beta\beta}$ means $a_{11} + a_{22} + a_{33}$

2) $x_i x_i$ or $x_\alpha x_\alpha$ means $x_1^2 + x_2^2 + x_3^2$

3) $a_{ij} b_{jk}$ means $a_{i1} b_{1k} + a_{i2} b_{2k} + a_{i3} b_{3k}$

4) $\dfrac{\partial u}{\partial x_j} \dfrac{\partial x_j}{\partial x_i'}$ means $\dfrac{\partial u}{\partial x_1} \dfrac{\partial x_1}{\partial x_i'} + \dfrac{\partial u}{\partial x_2} \dfrac{\partial x_2}{\partial x_i'} + \dfrac{\partial u}{\partial x_3} \dfrac{\partial x_3}{\partial x_i'}$

5) $T_{ijkl} S_{ij} V_k U_l$

$$= \underbrace{\sum_{ijkl} T_{ijkl} S_{ij} V_k U_l}_{\text{gets reduced to a scalar}}$$

- Repeated index, which gets summed over, is called a dummy index (does not matter what letter is used for it)

- Index that is not repeated is called a free index.

- When Einstein summation convention is being used, we are not warned by a summation symbol. Just inspect the expression & see which indices are repeated twice.

For example, we can write the transformation equations for a rank-four tensor as

$$T'_{\alpha\beta\gamma\delta} = a_{\alpha i} a_{\beta j} a_{\gamma k} a_{\delta l} T_{ijkl}$$

(sums understood over i,j,k,l)

Another example:

Suppose we set $\delta = \beta$, implying

$$T'_{\alpha\beta\gamma\beta} = a_{\alpha i} a_{\beta j} a_{\gamma k} a_{\beta l} T_{ijkl}$$

Let us focus on

$$a_{\beta j} a_{\beta l}$$

What does this simplify to?

recall $AA^T = I$

I.e., $a_{\beta j} a_{\beta l}$ is dot product of two different columns of a rotation matrix, so that it equals 1 if $j=l$ & equals 0 otherwise.

then $\alpha_{\beta j} \alpha_{\beta l} = \delta_{jl}$

then the whole expression simplifies as

$$a_{\alpha i}\, a_{\beta j}\, a_{\gamma k}\, a_{\beta l}\, T_{ijkl}$$

$$= a_{\alpha i}\, a_{\gamma k}\, \delta_{jl}\, T_{ijkl}$$

$$= a_{\alpha i}\, a_{\gamma k}\, T_{ijkj}$$

Conclude that T_{ijkj} are the components of a rank-two tensor. (2 free indices & 2 "a" factors are required)

<u>Contraction</u> is when we set 2 indices equal & sum. It always reduces the rank of a tensor by 2.

- In the example, we started w/ a rank-four tensor & contracted to a rank-two tensor.

- The dot product is another example of a contraction, Contracting the rank-two tensor U_iV_j to U_iV_i takes it to a scalar.

- Another example:

rank-3 tensor $T_{ij}V_k$ contracted to

$T_{ij}V_j = U_i$ goes to rank-one tensor like matrix-vector multiplication.

Symmetric + Antisymmetric tensors

Rank-two tensor is symmetric if $T_{ij} = T_{ji}$

+ antisymmetric if

$$T_{ij} = -T_{ji}$$

We can always write an arbitrary tensor (rank-two) as the summation of a symmetric + antisymmetric tensor

$$T_{ij} = \frac{1}{2}(T_{ij} + T_{ji}) + \frac{1}{2}(T_{ij} - T_{ji})$$

use similar terminology for higher rank tensors: if exchange of 2 indices leaves tensor component unchanged, then it is symmetric wrt. those indices.

If gives a minus sign,

then antisymmetriz