Lecture 1 - PHYS 4112

Topics to be covered in the course:

Ch. 10 - 13

Mathematical Methods in the Physical Sciences, 3rd Edition by Mary Boas (publisher Wiley)

Ch. 10 - Tensor Analysis

Ch. 11 - Special Functions

Ch. 12 - Series Solutions of Differential Equations

Ch. 13 - Partial Differential Equations

- We will use Mathematica in course, so please download from Tiger ware
- Grades are based on weekly assignments (homework) roughly 12 of them

Starting w/ Ch. 10 - Tensor Analysis

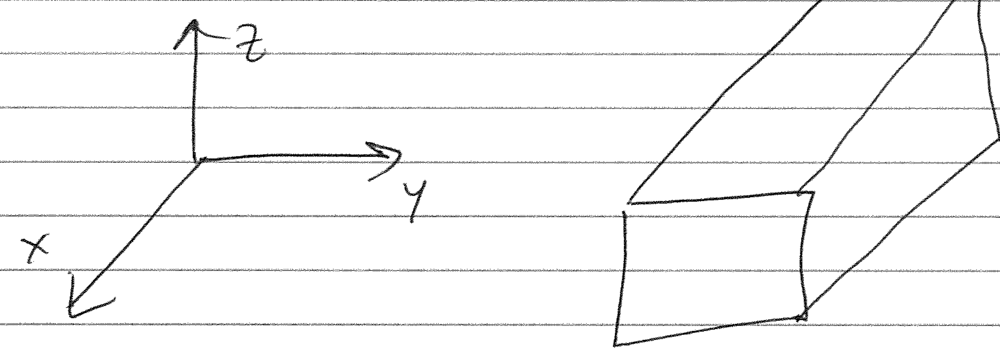
- tensors are mathematical objects w/ a rank (or order)

- a tensor of rank zero is a scalar
- a tensor of rank one is a vector
- a tensor of rank two is a matrix (but there is more to it than this)

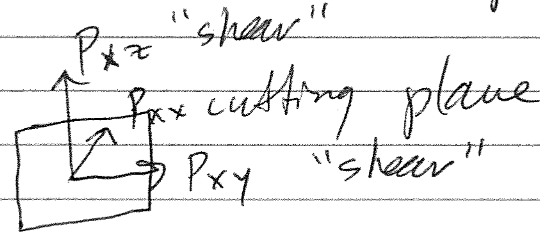
In 3D space, a scalar has $3^0 = 1$ component;
a vector has $3^1 = 3$ components
& a matrix has $3^2 = 9$ components
(rank-two tensor)
tensor of rank n in 3D space has 3^n components.

Example: Stress Tensor

A beam carrying a load has stresses & strains in the material of the beam.



cut a plane perpendicular to x direction. There is a pressure (force per unit area)



three components P_{xx}, P_{xy}, P_{xz}

we can make two other kinds of cutting planes, wrt y & z directions

Then there are the components

P_{yx}, P_{yy}, P_{yz}

& P_{zx}, P_{zy}, P_{zz}

can arrange these in a rank-two tensor as

$$\begin{bmatrix} P_{xx} & P_{xy} & P_{xz} \\ P_{yx} & P_{yy} & P_{yz} \\ P_{zx} & P_{zy} & P_{zz} \end{bmatrix}$$

on the diagonal are pressures or tensions

off diagonals are shear forces per unit area

Since force $= \int dA$ pressure, we would calculate

$$F_{yy} = \text{force} = \int_A P_{yy} \underbrace{dx\,dz}_{\text{integrating over perpendicular plane}}$$

Tensors are more than just a list of numbers. They have a physical meaning as well.
(are associated w/ physical quantities)

For an example, suppose that we attempt to represent a rotation as a vector.

We could say that

1) draw an arrow along the axis of rotation
2) make its length equal to the rotation angle in radians
3) direction of rotation is fixed by right-hand rule.

What is the problem w/ this?

we could take a book
rotate 90° about x axis &
90° about y axis

then repeat, but go
90° about y axis &
90° about x axis.
we end up in different positions.
But if we add the "rotation vectors"
we get the same vector in
both cases.
- So using vectors cannot capture
the physical essence of rotations.

To decide whether a physical
quantity can be described
by a vector, we should
be able to specify it w/
respect to one coordinate
system & figure out
what it is w/ respect to another
coordinate system

That is, we could ~~[illegible]~~ take combinations of ~~the vector~~ the components of the vector to realize the action of the vector.

For example: we could indicate displacements by a vector in a consistent way

$$(x,y,z) = (x,0,0) + (0,y,0) + (0,0,z)$$

we could realize the displacement (x,y,z) by first going x in x direction, then y in y direction, then z in z direction.

By assigning vectors to rotations as we did previously, this is not possible.

- With the stress tensor, we could cut a plane in any direction & figure out pressures & shear forces per units area as combinations of the components of the stress tensor.

- We could actually figure out a stress tensor in a different coordinate system by using the one that we gave.

Thus, all tensors that we employ in physics have

1) a physical meaning independent of the coordinate system considered &

2) there are simple mathematical rules for getting from its representation in one coordinate system to its representation in another.

What is the relationship between rank-one tensors & vectors?

- ideas of abstract vector spaces grew out of geometry of 3D displacement vectors.
- change of coordinate system corresponds to change of basis.

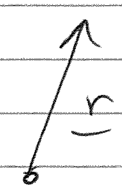
whether other physical quantities like force or stress can be modeled as rank-one tensors depends on whether they transform under a change of coordinate system in the same way that displacement vectors do.

A Cartesian tensor is one that transforms properly under a rotation of rectangular (x, y, z) axes.

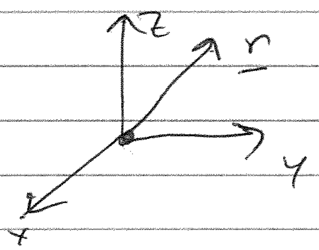
Focus for a while on Cartesian tensors.

Let us consider "passive rotations" in which the rank-one tensor is fixed but the axes are rotated.

Let $\underline{r}$ be a rank-one tensor (displacement vector)

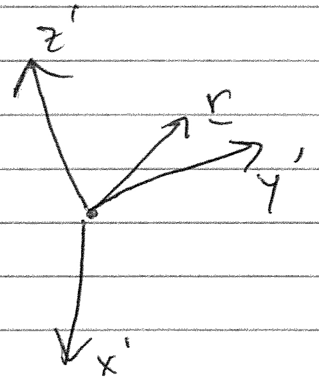


Relative to one coordinate system



it has components (x, y, z)

Relative to another, it has components (x', y', z')



We can then write out a list of the cosines of the angles between the different axes as

	x	y	z
x'	l_1	m_1	n_1
y'	l_2	m_2	n_2
z'	l_3	m_3	n_3

e.g., l_2 means the cosine of the angle between y' & x.

$\underline{r}$ can be written as (x, y, z) or (x', y', z')

What is the relationship between them? i.e., what is the transformation to get from (x, y, z) to (x', y', z')?

Let $\underline{i}\,\underline{j}\,\underline{k}$ denote the unit basis vectors for x, y, z &

let $\underline{i}'\,\underline{j}'\,\underline{k}'$ denote unit basis vectors for x', y', z'

Then

$$\underline{r} = \underline{i}x + \underline{j}y + \underline{k}z$$ this is the rank-one tensor

$$= \underline{i}'x' + \underline{j}'y' + \underline{k}'z$$

If we take the dot product of $\underline{r}$ w/ $\underline{i}'$, we get

$$\underline{r}\cdot\underline{i}' = \underline{i}\cdot\underline{i}'x + \underline{j}\cdot\underline{i}'y + \underline{k}\cdot\underline{i}'z$$

$$= x'$$

(b/c $\underline{i}'\cdot\underline{i}' = 1$, $\underline{j}'\cdot\underline{i}' = 0$, $\underline{k}'\cdot\underline{i}' = 0$)

$\underline{i}\cdot\underline{i}'$ is cosine of angle between $\underline{i}$ & $\underline{i}'$ (x & x' axes)

So $\underline{i}\cdot\underline{i}' = l_1$

Similarly $\underline{j}\cdot\underline{i}' = m_1$ & $\underline{k}\cdot\underline{i}' = n_1$

So that $x' = l_1 x + m_1 y + n_1 z$

Similarly, take dot product w/ $\underline{j}'$ + $\underline{k}'$ to get

$$y' = l_2 x + m_2 y + n_2 z$$

$$z' = l_3 x + m_3 y + n_3 z$$

These are the transformations equations for getting from (x, y, z) to (x', y', z')

We can do a similar exercise to find that

$$x = l_1 x' + l_2 y' + l_3 z'$$

$$y = m_1 x' + m_2 y' + m_3 z'$$

$$z = n_1 x' + n_2 y' + n_3 z'$$

We can then write the following matrix equation to describe the transformation

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

which ~~we~~ can abbreviate as

$$r' = A r$$

the other relation is

$$r = A^T r'$$

since going from the representation r to r' & back is equivalent to no change at all & this is true for all $\underline{r}$ we conclude that

$$A^T A = I \quad \text{or} \quad A^{-1} = A^T$$

& A is an orthogonal matrix.