Lecture 24

3/31/2011

noisy super-dense coding, state transfer, + trade-off coding

recall super-dense coding resource inequality

$$[q \to q] + [qq] \geq 2[c \to c]$$

by making channel noisy, we get entanglement-assisted classical comm.

$$\langle \mathcal{N} \rangle + H(A)[qq] \geq I(A;B)[c \to c]$$

What if we make the state noisy and allow access to noiseless qubit channels?

we get <u>noisy dense coding</u>:

$$H(A)[q \to q] + \langle \rho^{AB} \rangle \geq I(A;B)[c \to c]$$

will prove this in a way similar to EAC

and will get much more mileage out of it

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consider a state ρ^{AB} shared between Alice & Bob and take many copies $(\rho^{AB})^{\otimes n}$. Let state $|\psi\rangle^{ABR}$ be a purification of ρ^{AB}. For any $|\psi\rangle^{ABR}$, we can say it arises from the action of some isometry $U^{A' \to BR}$ acting on $|\psi\rangle^{AA'}$.

Setting then becomes closer to that of EAC.

Consider that many copies $(|\psi\rangle^{AA'})^{\otimes n}$ admits a type decomposition:

$$(|\psi\rangle^{AA'})^{\otimes n} = \sum_t \sqrt{p(t)} \underbrace{|\Phi_t\rangle^{A^n A'^n}}_{\text{maximally entangled states on orthogonal subspaces}}$$

Alice can then act w/ unitaries (as we had before in EAC) such that

$$U(s)^{A^n} |\psi\rangle^{A^n A'^n} = U^T(s)^{A'^n} |\psi\rangle^{A^n A'^n}$$

If she chooses at random, then

$$\overline{\rho}^{A^nB^n} \equiv \mathbb{E}_S \{ U(s)^{A^n} \rho^{A^nB^n} U^\dagger(s)^{A^n} \}$$

$$= \sum_t p(t)\, \pi_t^{A^n} \otimes \mathcal{N}^{A'^n \to B^n}(\pi_t^{A'^n})$$

where $\rho^{AB} = \mathcal{N}^{A' \to B}(\psi^{AA'})$

These unitaries are the basis for a random code, with quantum-assisted codewords of the form:

$$U^{A^n}(s(m))\, \rho^{A^nB^n}\, U^\dagger(s(m))^{A^n}$$

can find message subspace projectors & a total subspace projector such that

$$\Pi_m \equiv U(s)^{A^n}\, \Pi_{\rho,\delta}^{A^nB^n}\, U^\dagger(s)^{A^n}$$

$$\Pi \equiv \Pi_{\rho,\delta}^{A^n} \otimes \Pi_{\rho,\delta}^{B^n}$$

(similar to how we did for EAC) → can prove that the following conditions hold

1) $\mathrm{Tr}\{(\Pi_{\rho,\delta}^{A^n} \otimes \Pi_{\rho,\delta}^{B^n})(U(s)^{A^n} \rho^{A^nB^n} U^\dagger(s)^{A^n})\} \geq 1-\epsilon$

2) $\mathrm{Tr}\{(U(s)^{A^n} \Pi_{\rho,\delta}^{A^nB^n} U^\dagger(s)^{A^n})(U(s)^{A^n} \rho^{A^nB^n} U^\dagger(s)^{A^n})\} \geq 1-\epsilon$

3) $\mathrm{Tr}\{U(s)^{A^n} \Pi_{\rho,\delta}^{A^nB^n} U^\dagger(s)^{A^n}\} \leq 2^{n[H(AB)_\rho + \delta]}$

4) $\Pi \overline{\rho}^{A^nB^n} \Pi \leq 2^{-n[H(A)+H(B)-\delta]}\, \Pi$

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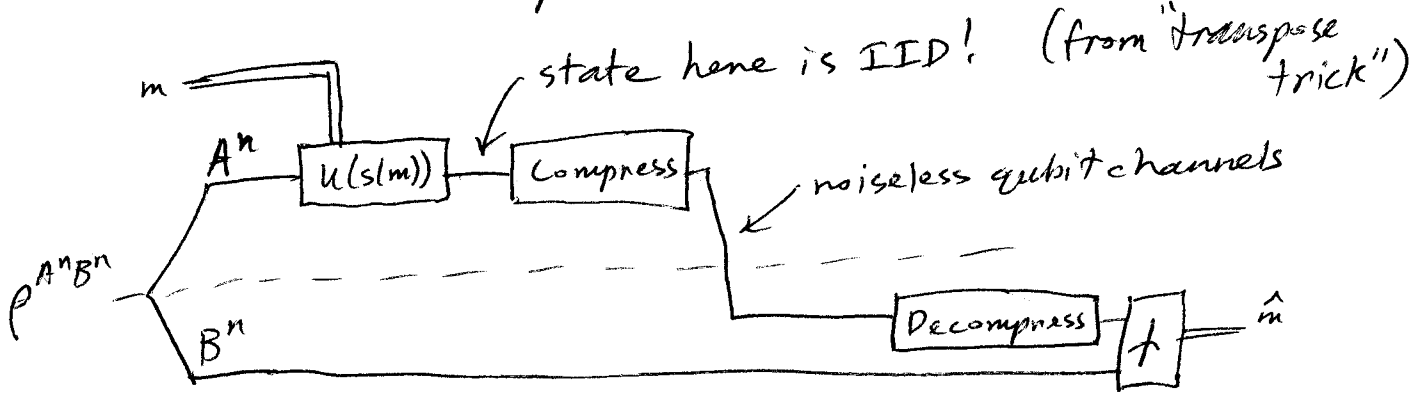
Thus packing lemma holds, giving us a code w/ low maximal prob. of error, as long as the message set size $|M| \approx 2^{n I(A;B)_\rho}$

How does protocol work?



by Schumacher compression & since state is IID

number of noiseless qubit channels required is $\approx nH(A)$. Thus, we get claimed resource inequality:

$$\langle \rho^{AB} \rangle + H(A)[q \to q] \geq I(A;B)[c \to c]$$

can upgrade to a protocol called coherent state transfer (similar to the way we upgraded EAC), by doing steps coherently.

Suppose Alice has the following state shared w/ reference:

$$|\psi\rangle^{R_1 A_1} \equiv \sum_{l,m} \alpha_{l,m} |l\rangle^{R_1} |m\rangle^{A_1}$$

She appends $|\psi\rangle^{R_1 A_1}$ to $|\varphi\rangle^{A^n B^n R^n}$ & does typ. subspace measurement on A^n:

$$|\psi\rangle^{R_1 A_1} \otimes |\varphi\rangle^{A^n B^n R^n}$$

Alice applies controlled unitary

$$\sum_m |m\rangle\langle m|^{A_1} \otimes U(s(m))^{A^n},$$

giving state

$$\sum_{l,m} \alpha_{l,m} |l\rangle^{R_1} |m\rangle^{A_1} U(s(m))^{A^n} |\varphi\rangle^{A^n B^n R^n}$$

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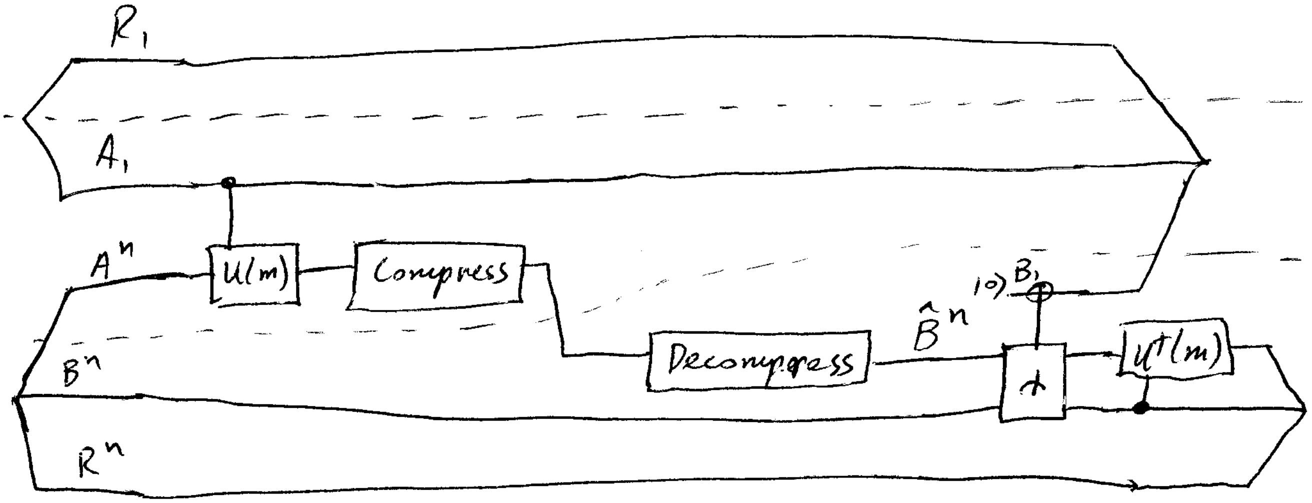

- Alice compresses
- Alice sends A^n over $nH(A)$ noiseless qubit channels
- Bob decompresses

Bob performs measurement coherently:

$$\sum_m \sqrt{\Lambda_m}^{\hat{B}^n B^n} \otimes |m\rangle^{B_1}$$

state is then close to

$$\sum_{l,m} \alpha_{l,m} |l\rangle^{R_1} |m\rangle^{A_1} \, U^{\hat{B}^n}(s(m)) |\varphi\rangle^{\hat{B}^n B^n R^n} |m\rangle^{B_1}$$

Bob performs controlled unitary

$$\sum_m |m\rangle\langle m|^{B_1} \otimes U^\dagger(s(m))^{\hat{B}}$$

& state becomes

$$\left(\sum_{l,m} \alpha_{l,m} |l\rangle^{R_1} |m\rangle^{A_1} |m\rangle^{B_1} \right) \otimes |\varphi\rangle^{\hat{B}^n B^n R^n}$$

So, Alice simulates $nI(A;B)_\rho$ coherent channels + also transfers her share of the state to Bob.

resource inequality is

$$\underbrace{\langle W^{S\to AB} : \rho^S \rangle}_{\text{initial state}} + H(A)_\rho \{q\to q\} \geq$$

$$I(A;B)_\rho \{q \to qq\} + \underbrace{\langle I^{S\to \hat{B}B} : \rho^S \rangle}_{\text{state transfer}}$$

can make a protocol for quantum-assisted transfer by

$$I(A;B)\{q\to qq\} = \tfrac{1}{2} I(A;B)\{q\to q\} + \tfrac{1}{2} I(A;B)\{qq\}$$

$$\therefore \boxed{\langle W^{S\to AB} : \rho^S \rangle + \tfrac{1}{2} I(A;R)\{q\to q\} \geq \tfrac{1}{2} I(A;B)\{qq\} + \langle I^{S\to \hat{B}B} : \rho^S \rangle}$$

"state merging"
"merging mother"
"FQSW"

combine further w/ teleportation to get

$$\langle W^{S\to AB} : \rho^S \rangle + \tfrac{1}{2} I(A;R)\{q\to q\} + I(A;R)\{c\to c\} + \tfrac{1}{2} I(A;R)\{qq\} \geq \tfrac{1}{2} I(A;B)\{qq\} + \langle I^{S\to \hat{B}B} : \rho^S \rangle + \tfrac{1}{2} I(A;R)\{q\to q\}$$

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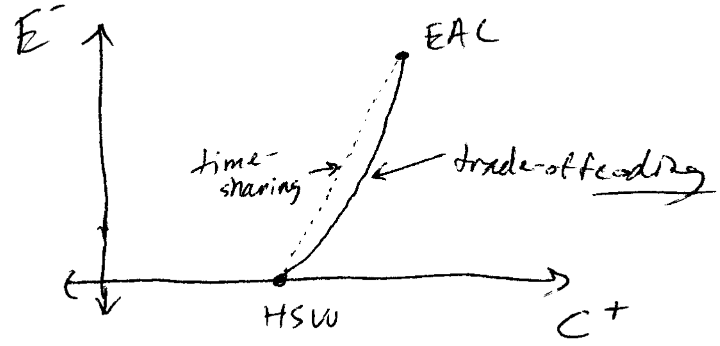
$\therefore$

$$\langle W^{S\to AB} : \rho^S \rangle + I(A;R)\{c\to c\} \geq I(A\rangle B)\{qq\} + \langle I^{S\to \hat{B}B} : \rho^S \rangle$$

if $I(A\rangle B) > 0$, produces entanglement,
if $I(A\rangle B) < 0$, consumes entanglement.

Trade-off Coding

- a way to combine HSW & EAC in order to beat time-sharing (for some channels)
- consider trade-off between HSW & EAC



Can achieve the resource inequality

$$\langle \mathcal{N} \rangle + H(A|X)\,[qq] \geq I(AX;B)\,[c \to c]$$

where entropies are w/ respect to

$$\sum_x p_X(x)\, |x\rangle\langle x|^X \otimes \mathcal{N}^{A' \to B}(\phi_x^{AA'})$$

special cases: if $\phi_x^{AA'} = \phi^A \otimes \varphi_x^{A'}$ then HSW.

if $p_X(x) = \delta_{x,0}$ & $\phi_0^{AA'} = \phi_*^{AA'}$ for EAC, then EAC.

in between is a trade-off code.

How ~~to~~ achieve?

Consider HSW code

$$\{\rho_{x^n(m)}\}_{m \in \mathcal{M}}$$

where $\rho_{x^n(m)} = \rho_{x_1(m)} \otimes \rho_{x_2(m)} \otimes \cdots \otimes \rho_{x_n(m)}$

There is some decoding POVM $\{\Lambda_m^{B^n}\}$ such that

$$\mathrm{Tr}\{\Lambda_m^{B^n} \mathcal{N}^{A'^n \to B^n}(\rho_{x^n(m)}^{A'^n})\} \geq 1-\epsilon$$

as long as $|\mathcal{M}| \approx 2^{nI(X;B)}$

Since we choose each sequence x^n to be strongly typical, we can permute it so that

$$\pi(x^n) = \underbrace{a_1 \cdots a_1}_{np_X(a_1)} \; \underbrace{a_2 \cdots a_2}_{np_X(a_2)} \cdots \underbrace{a_{|\mathcal{X}|} \cdots a_{|\mathcal{X}|}}_{np_X(a_{|\mathcal{X}|})}$$

same for the quantum codewords

$$\pi(\rho_{x^n}) = \underbrace{\rho_{a_1} \otimes \cdots \otimes \rho_{a_1}}_{np_X(a_1)} \otimes \cdots \otimes \underbrace{\rho_{a_{|\mathcal{X}|}} \otimes \cdots \otimes \rho_{a_{|\mathcal{X}|}}}_{np_X(a_{|\mathcal{X}|})}$$

now consider a purification $\varphi_x^{AA'}$ of each $\rho_x^{A'}$. we get purified codewords

$$\varphi_{x^n(m)}^{A^nA'^n} \equiv \varphi_{x_1(m)}^{A_1A_1'} \otimes \cdots \otimes \varphi_{x_n(m)}^{A_nA_n'}$$

Apply same permutation to these ~~to~~ get

$$\pi(\ell_{x^n}) = \underbrace{\ell_{a_1} \otimes \cdots \otimes \ell_{a_1}}_{n p_X(a_1)} \otimes \cdots \otimes \underbrace{\ell_{a_{|X|}} \otimes \cdots \otimes \ell_{a_{|X|}}}_{n p_X(a_{|X|})}$$

we now get a strategy for trade-off coding

Alice begins w/ some standard sequence $\hat{x}^n$ in lexicographical order. She then prepares her shares of the states in the order above. Since each block is large enough for law of large numbers to come into play, there exists an EAC code of size $\approx 2^{n\, I(A;B)_{\ell(\ell_{a_i})}}$ for each block w/ length $n p_X(a_i)$.

these codewords have the property that state is a tensor-power state on each block after tracing over Bob's half of the entanglement. Alice then permutes so that state is $\rho_{x^n(m)}$ after tracing over Bob.

Decoder 1st performs HSW measurement to determine codeword $\rho_{x^n(m)}$ (measurement does not act on Bob's half of entanglement). After knowing $x^n(m)$, Bob permutes his systems to be in lexicographical order & performs EAC decoding POVM for each block of size $n p_X(a_i)$

How does rate add up to $I(AX;B)$?

from chain rule, we know that

$$n I(AX;B) = n I(X;B) + n I(A;B|X)$$

we get $n I(X;B)$ bits from HSW

for each block of size $n p_X(a_i)$, we get $n p_X(a_i) I(A;B)_{N(\rho_{a_i})}$ bits.

Adding up the blocks gives

$$\sum_i n p_X(a_i) I(A;B)_{N(\rho_{a_i})} = n I(A;B|X) \text{ bits}$$

similarly, ~~the~~ ent. cons. rate is $H(A|X)$

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we can make EAC part coherent. this gives

$$\langle N \rangle + H(A|X)\{qq\} \geq I(A;B|X)\{q \to qq\} + I(X;B)\{c \to c\}$$

via coherent comm. identity, we get

$$\langle N \rangle + \tfrac{1}{2}I(A;E|X)\{qq\} \geq \tfrac{1}{2}I(A;B|X)\{q \to q\} + I(X;B)\{c \to c\}.$$

EA classical + quantum comm.

combine this w/ teleportation, dense coding, + entanglement distribution to get

$$\begin{bmatrix} C \\ Q \\ E \end{bmatrix} = \begin{bmatrix} -2 & 2 & 0 \\ 1 & -1 & -1 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} + \begin{bmatrix} I(X;B) \\ \tfrac{1}{2}I(A;B|X) \\ -\tfrac{1}{2}I(A;E|X) \end{bmatrix}$$

teleportation, dense coding, entanglement distribution

using $\alpha, \beta, \gamma \geq 0$ + inverting gives quantum dynamic capacity region

$$\boxed{\begin{aligned} C + 2Q &\leq I(AX;B) \\ Q + E &\leq I(A\rangle BX) \\ C + Q + E &\leq I(X;B) + I(A\rangle BX) \end{aligned}}$$

C, Q, E are positive or negative