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Lecture 21

1

Entanglement-Assisted Classical Comm.

super-dense coding:

$$\{q \to q\} + \{qq\} \geq 2\{c \to c\}$$

what if channel becomes noisy?

$$\langle \mathcal{N} \rangle + E\{qq\} \geq C\{c \to c\}$$

rate of entanglement consumption

rate of classical comm.

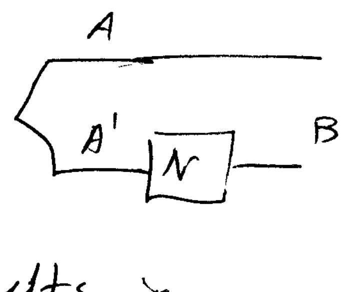
if entanglement unlimited, then very nice answer:

$$\sup\{C : C \text{ is achievable}\} = I(\mathcal{N})$$

where

$$I(\mathcal{N}) = \max_{\phi^{AA'}} I(A;B)_\rho$$

$$\rho = \mathcal{N}^{A' \to B}(\phi^{AA'})$$



one of strongest known results in quantum Shannon theory

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general information processing task:

(n, C, ϵ) EA code

begin with some entangled state

$$|\Phi\rangle^{T_A T_B} \equiv \frac{1}{\sqrt{D}} \sum_{i=0}^{D-1} |i\rangle^{T_A} |i\rangle^{T_B}$$

Alice selects $m \in \mathcal{M}$ + performs

$$\mathcal{E}_m^{T_A \to A'^n}(\Phi^{T_A T_B})$$

transmits systems A'^n over many channel uses

$$\omega_m^{T_B B^n} = \mathcal{N}^{A' \to B^n}(\mathcal{E}_m^{T_A \to A'^n}(\Phi^{T_A T_B}))$$

Bob performs some POVM $\{\Lambda_m^{T_B B^n}\}$ to detect. probability of correct detection

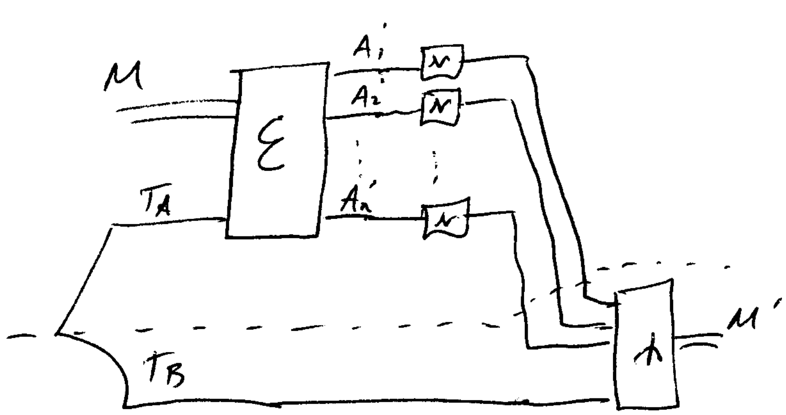
$$\Pr\{M' = m \mid M = m\} = \mathrm{Tr}\{\Lambda_m^{T_B B^n} \omega_m^{T_B B^n}\}$$

prob. of error $p_e(m) = \mathrm{Tr}\{(I - \Lambda_m^{T_B B^n}) \omega_m^{T_B B^n}\}$

$$p_e^* = \max_m p_e(m)$$

rate $C = \frac{\log_2 |\mathcal{M}|}{n}$

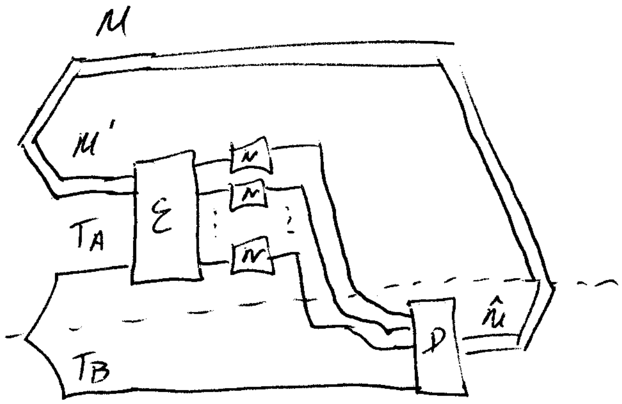
want $p_e^* \leq \epsilon$



Begin by proving converse theorem.

- Suppose instead that they are trying to use the channel for common randomness generation.

- capacity for this task can only be larger than that for classical comm.



$$\overline{\Phi}^{MM'} \equiv \frac{1}{|M|} \sum_m^M |m\rangle\langle m| \otimes |m\rangle\langle m|$$

Initial state: $\overline{\Phi}^{MM'} \otimes \Phi^{T_A T_B}$

then $\omega^{M T_B B^n} = \mathcal{N}^{A'^n \to B^n}\left(\mathcal{E}^{M' T_A \to A'^n}\left(\overline{\Phi}^{MM'} \otimes \Phi^{T_A T_B}\right)\right)$

↑ this is classical-quantum state

Finally, Bob decodes

$$(\omega')^{M\hat{M}} \equiv \mathcal{D}^{T_B B^n \to \hat{M}}\left(\omega^{M T_B B^n}\right)$$

For a good code, the following should hold

$$\left\| \overline{\Phi}^{M\hat{M}} - (\omega')^{M\hat{M}} \right\|_1 \leq \epsilon$$

$$
\begin{aligned}
nC &= H(M)_{\overline{\Phi}} \\
&= H(M)_{\overline{\Phi}} - H(M|\hat{M})_{\overline{\Phi}} \\
&= I(M;\hat{M})_{\overline{\Phi}} \\
&\leq I(M;\hat{M})_{\omega'} + n\epsilon' \qquad \text{(Alicki-Fannes')} \\
&\leq I(M;B^n T_B)_\omega + n\epsilon' \qquad \text{(QDP)} \\
&= I(T_B M;B^n)_\omega + I(M;T_B)_\omega - I(B^n;T_B)_\omega + n\epsilon' \\
&= I(T_B M;B^n)_\omega - I(B^n;T_B)_\omega + n\epsilon' \qquad \text{(chain rule)} \; (I(M;T_B)_\omega = 0) \\
&\leq I(T_B M;B^n)_\omega + n\epsilon'
\end{aligned}
$$

state $\omega^{MT_BB^n}$ is a classical-quantum state of the form

$$\omega^{MT_BB^n} = \frac{1}{|M|}\sum_m |m\rangle\langle m|^M \otimes \omega_m^{T_BB^n}$$

more general state is

$$\rho^{XAB^n} = \sum_x p(x)\, |x\rangle\langle x|^X \otimes \mathcal{N}^{A'^n\to B^n}(\rho_x^{AA'^n})$$

So,

$$\leq \max_{\rho^{XAA'^n}} I(AX;B^n)_\rho + n\epsilon'$$

can strengthen this result considerably

1st, consider state

$$\rho^{XAA'^n} \equiv \sum_x p(x) |x\rangle\langle x|^X \otimes \rho_x^{AA'^n}$$

diagonalize each $\rho_x^{AA'^n}$ as

$$\rho_x^{AA'^n} = \sum_y p(y|x) \psi_{x,y}^{AA'^n}$$ +

augment above state to be

$$\rho^{XYAA'^n} = \sum_{x,y} p(x)p(y|x) |x\rangle\langle x|^X \otimes |y\rangle\langle y|^Y \otimes \psi_{x,y}^{AA'^n}$$

then, by QDP, it holds that

$$I(AX;B^n) \leq I(A\underline{XY};B^n)$$

consider as a single classical system

thus, it is sufficient to optimize over cq states where conditional state is pure. So

$$\max_{\{\text{mixed cq}\}} I(AX;B^n) = \max_{\{\text{pure cq}\}} I(AX;B^n) = \max_{\phi^{AA'^n}} I(A;B^n)$$

by homework result

Then $\max_{\phi_{AA'^n}} I(A;B^n) = n \max_{\phi_{AA'}} I(A;B)$

so we get, $nC \le n\, I(\mathcal{N}) + n\epsilon'$ for quantum mutual information

Let's prove this last statement.

We want to show that

$$I(\mathcal{N}_1 \otimes \mathcal{N}_2) = I(\mathcal{N}_1) + I(\mathcal{N}_2)$$

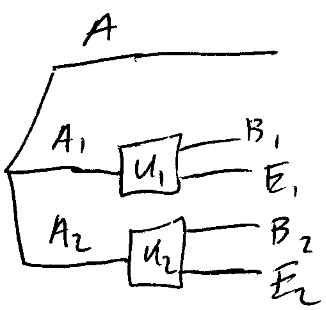
for any two channels $\mathcal{N}_1$ + $\mathcal{N}_2$.

Know that $I(\mathcal{N}_1 \otimes \mathcal{N}_2) \ge I(\mathcal{N}_1) + I(\mathcal{N}_2)$

Thus, should prove that

$$I(\mathcal{N}_1 \otimes \mathcal{N}_2) \le I(\mathcal{N}_1) + I(\mathcal{N}_2)$$

Consider



$$\begin{aligned}
I(\mathcal{N}_1 \otimes \mathcal{N}_2) &= I(A; B_1 B_2) \quad \text{for some state} \\
&= H(A) + H(B_1 B_2) - H(A B_1 B_2) \\
&= H(B_1 B_2 E_1 E_2) + H(B_1 B_2) - H(E_1 E_2) \\
&= H(B_1 B_2 | E_1 E_2) + H(B_1 B_2) \\
(\text{SSA} + \text{SA}) \quad &\le H(B_1|E_1) + H(B_2|E_2) + H(B_1) + H(B_2) \\
&= \{H(B_1) + H(B_1 E_1) - H(E_1)\} + \{H(B_2) + H(B_2 E_2) - H(E_2)\}
\end{aligned}$$

$$= \left[H(B_1) + H(AA_2) - H(AA_2B_1)\right] + \left[H(B_2) + H(AA_1) - H(AA_1B_2)\right]$$

$$= I(AA_2; B_1) + I(AA_1; B_2)$$

$$\leq I(N_1) + I(N_2)$$

By induction, it follows that

$$I(N^{\otimes n}) = n\, I(N)$$

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compute EA capacity of erasure channel. Recall it is

$$\rho^{A'} \to (1-\epsilon)\, \rho^B + \epsilon\, |e\rangle\langle e|^B$$

send half of a bipartite state $\phi^{AA'}$ through erasure channel:

$$\phi^{AA'} \to \sigma^{AB} \equiv (1-\epsilon)\, \phi^{AB} + \epsilon\left(\phi^A \otimes |e\rangle\langle e|^B\right)$$

perform an isometry on B that will make things a bit clearer:

$$U^{B \to BX} \equiv \Pi^B \otimes |0\rangle^X + |e\rangle\langle e|^B \otimes |1\rangle^X$$

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Bob performs isometry:

$$\sigma^{ABX} = \left(\mathcal{U}^{B\to BX}\right)\sigma^{AB}\left(\mathcal{U}^{B\to BX}\right)^{\dagger}$$

$$= (1-\epsilon)\,\phi^{AB}\otimes|0\rangle\langle 0|^{X} + \epsilon\left[\phi^{A}\otimes|e\rangle\langle e|^{B}\otimes|1\rangle\langle 1|^{X}\right]$$

Then $I(A;BX)_\sigma = I(A;B)_\sigma$ b/c $\mathcal{U}$ is isometry + entropies don't change

$$I(A;BX)_\sigma = H(A)_\sigma + H(BX)_\sigma - H(ABX)_\sigma$$

$$= H(A)_\sigma + H(B|X)_\sigma - H(AB|X)_\sigma$$

$$= H(A)_\phi + \left[(1-\epsilon)\left[H(B)_\phi - H(AB)_\phi\right]\right] + \epsilon\left[H(B)_{|e\rangle} - H(AB)_{\phi\otimes|e\rangle\langle e|}\right]$$

$$= H(A)_\phi + (1-\epsilon)\left\{H(A)_\phi\right\} + \epsilon\left[-H(A)_\phi\right]$$

$$= 2(1-\epsilon)\,H(A)_\phi$$

choose ϕ to be maximally entangled, giving

$$2(1-\epsilon)\log d_A$$

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strategy for achieving capacity is easy. just do super-dense coding on all channel uses.

For a fraction $1-\epsilon$, strategy works (and they know it does – turns out feedback does not improve EA capacity).

due to Bowen 0209076