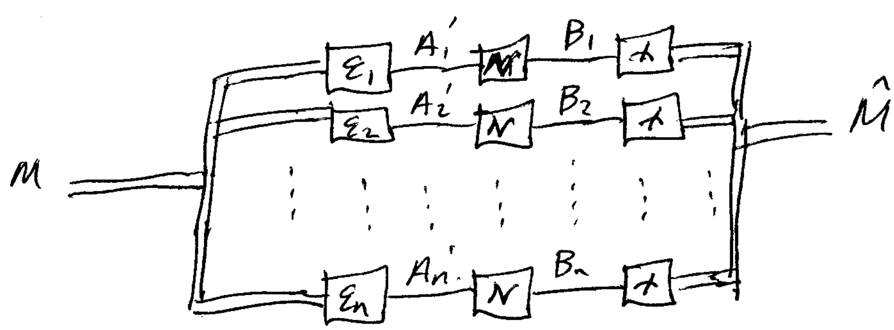
Lecture 19

3/15/2011

Classical Communication over a Quantum Channel

- first naive approach is to use a measurement at every channel output

scheme is



I.e., Alice chooses codewords from ensemble $\{p(x), \rho_x\}$, generating

$$\rho_{x^n(m)} \equiv \rho_{x_1(m)} \otimes \rho_{x_2(m)} \otimes \cdots \otimes \rho_{x_n(m)}$$

channel output is

$$\mathcal{N}(\rho_{x_1(m)}) \otimes \mathcal{N}(\rho_{x_2(m)}) \otimes \cdots \otimes \mathcal{N}(\rho_{x_n(m)})$$

Bob performs some POVM at individual channel outputs (POVM is $\{\Lambda_y\}$)

Induces classical channel

$$p_{Y_1 \cdots Y_n | X_1 \cdots X_n}(y_1 \cdots y_n | x_1(m) \cdots x_n(m)) = \mathrm{Tr}\{\mathcal{N}^{\otimes n}(\rho_{x^n(m)})(\Lambda_{y_1} \otimes \cdots \otimes \Lambda_{y_n})\}$$

3/15/2011

$$= \mathrm{Tr}\left\{ \left[\mathcal{N}(\rho_{x_1(m)}) \otimes \cdots \otimes \mathcal{N}(\rho_{x_n(m)}) \right] \left[\Lambda_{y_1} \otimes \cdots \otimes \Lambda_{y_n} \right] \right\}$$

$$= \prod_{i=1}^{n} \mathrm{Tr}\left\{ \mathcal{N}(\rho_{x_i(m)}) \Lambda_{y_i} \right\}$$

just many instances of IID channel:

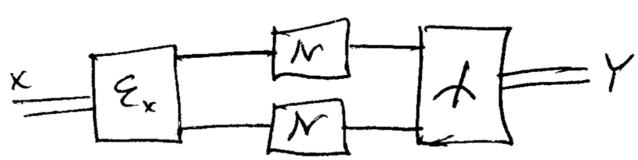
$$p_{Y|X}(y|x) \equiv \mathrm{Tr}\left\{ \mathcal{N}(\rho_x) \Lambda_y \right\}$$

can achieve rate

$$I_{acc}(\mathcal{N}) \equiv \max_{\{p_X(x), \rho_x, \Lambda\}} I(X;Y)$$

by Shannon's theorem ...

strategy is not necessarily optimal. consider inducing a classical channel from



accessible information of this "blocked" channel is

$$\frac{1}{2} I_{acc}(\mathcal{N} \otimes \mathcal{N}) \geq I_{acc}(\mathcal{N})$$

can keep blocking strategies to achieve

$$I_{reg,acc}(\mathcal{N}) \equiv \lim_{k \to \infty} \frac{1}{k} I_{acc}(\mathcal{N}^{\otimes k})$$

3/15/2011

the characterization of classical capacity w/ regularized accessible information is completely useless...

cannot compute it in the limit

we know that a good upper bound on accessible information is Holevo ~~information~~ of channel:

$$I_{acc}(\mathcal{N}) \leq \chi(\mathcal{N}) \equiv \max_{\rho} I(X;B)$$

where

$$\rho^{XB} \equiv \sum_x p_X(x) \, |x\rangle\langle x|^X \otimes \mathcal{N}^{A'\to B}(\psi_x^{A'})$$

Holevo information of channel is a bit nicer analytically. It is <u>concave</u> in input ensemble + <u>pure states</u> at channel input are sufficient to attain maximum.

<u>Concavity</u>: consider ensemble $\{p(x_1), \rho_{x_1}^{A'}\}$
+ ensemble $\{p(x_2), \sigma_{x_2}^{A'}\}$

"mixed" ensemble is $\{(\lambda p(x_1), \rho_{x_1}^{A'}), ((1-\lambda)p(x_2), \sigma_{x_2}^{A'})\}$

c-q state for 1st ensemble is

$$\rho^{XB} \equiv \sum_{x_1} p(x_1) |x_1\rangle\langle x_1|^X \otimes \mathcal{N}(\rho_{x_1}^{A'})$$

c-q state for 2nd ensemble is

$$\sigma^{XB} \equiv \sum_{x_2} p(x_2) |x_2\rangle\langle x_2|^X \otimes \mathcal{N}(\sigma_{x_2}^{A'})$$

c-q state for mixed ensemble is

$$\omega^{UXB} \equiv \lambda |0\rangle\langle 0|^U \otimes \rho^{XB} + (1-\lambda)|1\rangle\langle 1|^U \otimes \sigma^{XB}$$

concavity would mean that

$$\lambda I(X;B)_\rho + (1-\lambda) I(X;B)_\sigma \leq I(UX;B)_\omega$$

this follows from chain rule for QMI:

$$\begin{aligned} I(UX;B)_\omega &= I(X;B|U)_\omega + I(U;B)_\omega \\ &= \lambda I(X;B)_\rho + (1-\lambda) I(X;B)_\sigma + I(U;B)_\omega \\ &\geq \lambda I(X;B)_\rho + (1-\lambda) I(X;B)_\sigma \end{aligned}$$

pure states are sufficient for optimizing Holevo info:

Consider arbitrary c-q state

$$\rho^{XB} \equiv \sum_x p(x)\, |x\rangle\langle x|^X \otimes \mathcal{N}(\rho_x^{A'})$$

can diagonalize each $\rho_x^{A'}$ as

$$\rho_x^{A'} = \sum_y p(y|x)\, \psi_{x,y}^{A'}$$

these are pure

form augmented state

$$\rho^{XYB} = \sum_{x,y} p(x)p(y|x)\, |x\rangle\langle x|^X \otimes |y\rangle\langle y|^Y \otimes \mathcal{N}(\psi_{x,y}^{A'})$$

observe that $\mathrm{Tr}_Y\{\rho^{XYB}\} = \rho^{XB}$

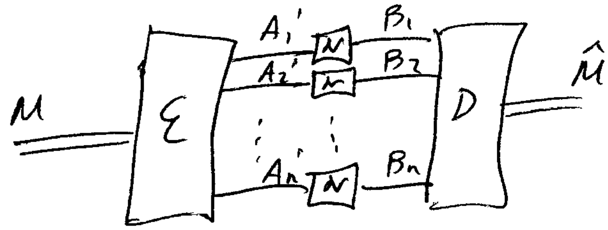
Also

$$I(X;B)_\rho \leq I(XY;B)_\rho \qquad (QDP)$$

Consider XY as joint classical register. Thus, pure states are sufficient

We would like to show that Holevo info. is <u>achievable</u>

<u>Info. Processing Task</u>



Alice selects message $m \in \mathcal{M}$

inputs some state $\rho_m^{A'^n}$ to channel $\mathcal{N}^{\otimes n}$.

state is $\mathcal{N}^{\otimes n}(\rho_m^{A'^n})$

Bob has some detection POVM $\{\Lambda_m\}$

probability of correct detection is

$$\Pr\{\hat{M} = m \mid M = m\} = \mathrm{Tr}\{\Lambda_m \mathcal{N}^{\otimes n}(\rho_m^{A'^n})\}$$

probability of incorrect detection is then

$$p_e(m) = 1 - \Pr\{\hat{M} = m \mid M = m\}$$
$$= \mathrm{Tr}\{(I - \Lambda_m)\mathcal{N}^{\otimes n}(\rho_m^{A'^n})\}$$

$$p_e^* \equiv \max_{m \in \mathcal{M}} p_e(m)$$

rate $R = \frac{1}{n}\log_2|\mathcal{M}|$

rate R is <u>achievable</u> if $\exists$ (n, R, ϵ) code $\forall \epsilon > 0$ + suff. large n.

we will prove that

$$\max_{\rho} I(X;B) \quad \text{w.r.t.}$$

$$\rho^{XB} \equiv \sum_x p(x)\,|x\rangle\langle x|^X \otimes \mathcal{N}^{A'\to B}(\rho_x^{A'})$$

is an achievable rate for classical comm.

First, we require the tool of conditional quantum typicality (Sect. 13.9 + 14.2.3)

suppose distribution $p_X(0) = 1/4$, $p_X(1) = 1/4$, $p_X(2) = 1/2$

could generate

20102010 2212

can then draw from 3 different quantum information sources based on the above sequence and generate a tensor product state

$\rho_2 \otimes \rho_0 \otimes \rho_1 \otimes \rho_0 \otimes \rho_2 \otimes \rho_0 \otimes \cdots$

can permute these states as

$$\underbrace{\rho_0^{\otimes 4}} \otimes \underbrace{\rho_1^{\otimes 3}} \otimes \underbrace{\rho_2^{\otimes 5}}$$

there is a different typical projector for each of these blocks

3/15/2011

cond. typ. projector: $\Pi_{\rho_0}^{4} \otimes \Pi_{\rho_1}^{3} \otimes \Pi_{\rho_2}^{5}$

can then permute back as

$$\Pi_{\rho_0}^{I_0} \otimes \Pi_{\rho_1}^{I_1} \otimes \Pi_{\rho_2}^{I_2}$$

where sets I_x keep track of on which systems projectors act.

For our example,

$$I_0 = \{2, 4, 6, 8\}$$
$$I_1 = \{3, 7, 11\}$$
$$I_2 = \{1, 5, 9, 10, 12\}$$

In the general case, we can have state

$$\rho_{x^n} = \rho_{x_1} \otimes \rho_{x_2} \otimes \cdots \otimes \rho_{x_n}$$

where $x_i \in \mathcal{X}$

if we choose x^n to be strongly typical, then there are roughly $n p_X(a_i)$ occurrences of symbol $a_i \in \mathcal{X}$, so we can permute sequence as

$$\pi(\rho_{x^n}) = \underbrace{\rho_{a_1} \otimes \cdots \otimes \rho_{a_1}}_{\substack{n p_X(a_1) \\ \text{systems}}} \otimes \cdots \otimes \underbrace{\rho_{a_{|\mathcal{X}|}} \otimes \cdots \otimes \rho_{a_{|\mathcal{X}|}}}_{\substack{n p_X(a_{|\mathcal{X}|}) \\ \text{systems}}}$$

then conditionally typical projector is

$$\underbrace{\Pi^{B^n p_X(a_1)}_{\rho_{a_1},\delta}}_{\text{typical projector for } \rho_{a_1}} \otimes \cdots \otimes \underbrace{\Pi^{B^n p_X(a_{|\mathcal{X}|})}_{\rho_{a_{|\mathcal{X}|}},\delta}}_{\text{typ. projector for } \rho_{a_{|\mathcal{X}|}}}$$

permute back to get

$$\Pi^{B^n|X^n}_{\rho_{x^n},\delta} = \bigotimes_{i=1}^{|\mathcal{X}|} \Pi^{I_{a_i}}_{\rho_{a_i},\delta}$$

where I_{a_i} are index sets

cond. typ. projector has several nice properties:

1) $\mathrm{Tr}\{\Pi^{B^n|X^n}_{\rho_{x^n},\delta}\, \rho_{x^n}\} \geq 1-\epsilon$ (LLN)

2) $2^{n[H(B|X)-\delta']} \leq \mathrm{Tr}\{\Pi^{B^n|X^n}_{\rho_{x^n}}\} \leq 2^{n[H(B|X)+\delta']}$

3) $2^{-n[H(B|X)+\delta']}\, \Pi^{B^n|X^n}_{\rho_{x^n},\delta} \leq \Pi^{B^n|X^n}_{\rho_{x^n},\delta}\, \rho_{x^n}\, \Pi^{B^n|X^n}_{\rho_{x^n},\delta} \leq$

$2^{-n[H(B|X)-\delta']}\, \Pi^{B^n|X^n}_{\rho_{x^n},\delta}$

can also prove that $\mathrm{Tr}\{\Pi^{B^n}_{\rho,\delta}\, \rho_{x^n}\} \geq 1-\epsilon$

where $\rho = \sum_x p(x)\, \rho_x$

3/15/2011

scheme for a random code:

1) For each message m, Alice chooses a sequence x^n independent of the message according to distribution

$$p_{X'^n}(x^n) = \begin{cases} \left[\sum_{x^n \in \mathcal{T}^{X^n}} p_{X^n}(x^n)\right]^{-1} p_{X^n}(x^n) & : x^n \in \mathcal{T}_\delta^{X^n} \\ 0 & : x^n \notin \mathcal{T}_\delta^{X} \end{cases}$$

this ensures that sequences x^n are strongly typical $\left(\frac{N(x|x^n)}{n} \approx p_X(x) \right)$

each sequence x^n then leads to a quantum codeword

$$\rho_{x^n} = \rho_{x_1} \otimes \rho_{x_2} \otimes \cdots \otimes \rho_{x_n}$$

if we would like, we can denote explicitly the association of the message m w/ sequence x^n as $x^n(m)$. Do this for every message m to get a codebook $\{\rho_{x^n(m)}\}_{m \in \mathcal{M}}$

leads to states

$$\sigma_{x^n(m)} = \mathcal{N}^{\otimes n}(\rho_{x^n(m)})$$

for Bob

2) We need to have a detection POVM for Bob. Consider $\{\Lambda_m\}_{m\in\mathcal{M}}$ where

$$\Lambda_m \equiv \left(\sum_{m'=1}^{|\mathcal{M}|} \Pi^{B^n|X^n}_{\sigma_{x^n(m')},\delta}\right)^{-1/2} \Pi^{B^n|X^n}_{\sigma_{x^n(m)},\delta} \left(\cdot \right)^{-1/2}$$

Is this a POVM? (need positivity + resolution of identity)

positivity) $\Pi^{B^n|X^n}_{\sigma_{x^n},\delta} \geq 0$

$\therefore$

$$(\cdot)^{-1/2}\, \Pi^{B^n|X^n}_{\sigma_{x^n(m)},\delta}\, (\cdot)^{-1/2} \geq 0$$

resolution of Identity)

$$\sum_m (\cdot)^{-1/2}\, \Pi^{B^n}_{\sigma_{x^n(m)},\delta}\, (\cdot)^{-1/2}$$

$$= (\cdot)^{-1/2}\left(\sum_m \Pi^{B^n}_{\sigma_{x^n(m)},\delta}\right)(\cdot)^{-1/2} \quad (*)$$

$A^{-1/2}$ is defined as $\sum_i (a_i)^{-1/2}|i\rangle\langle i|$ for $A = \sum_i a_i |i\rangle\langle i|$

So, $(*) = I$ + this is a valid POVM.

can prove achievability w/ four conditions

1) $\mathrm{Tr}\{\Pi^{B^n|x^n}_{\sigma_{x^n},\delta}\ \sigma^{B^n}_{x^n}\} \geq 1-\epsilon$

2) $\mathrm{Tr}\{\Pi^{B^n}_{\sigma,\delta}\ \sigma^{B^n}_{x^n}\} \geq 1-\epsilon$

3) $\mathrm{Tr}\{\Pi^{B^n|x^n}_{\sigma_{x^n},\delta}\} \leq 2^{n[H(B|X)+\delta]}$

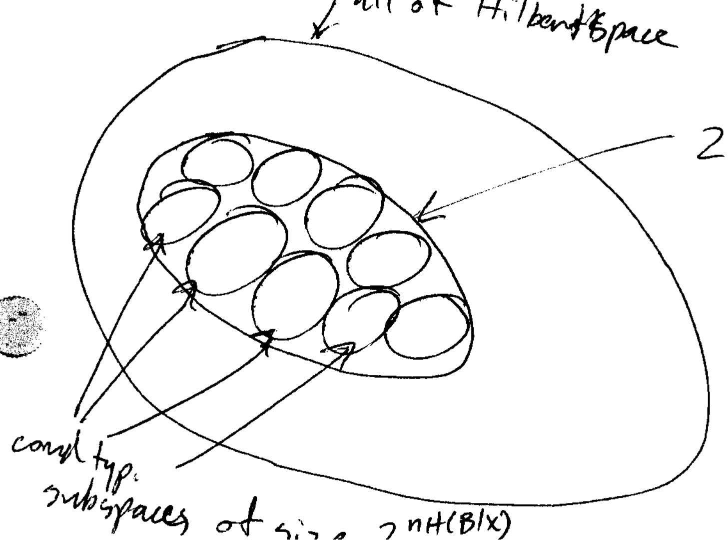
4) $\Pi^{B^n}_{\sigma,\delta}\ \sigma^{\otimes n}\ \Pi^{B^n}_{\sigma,\delta} \leq 2^{-n[H(B)-\delta]}$

where $\sigma^{\otimes n} = \sum_{x^n} p(x^n)\,\sigma_{x^n}$

intuition: follow the probability w/ typical subspaces...

if Bob doesn't know x^n, his state is $\sigma^{\otimes n}$ w/ typical projector $\Pi^{B^n}_{\sigma}$ of size $2^{nH(B)}$

if Bob knows x^n, state is σ_{x^n} w/ cond. typical proj. of size $2^{nH(B|X)}$



should be able to pack in $\dfrac{2^{nH(B)}}{2^{nH(B|X)}}$ messages $\approx 2^{nI(X;B)}$