3/10/2011 $\boxed{\text{Lecture 18}}$

Example of Schumacher Compression

Suppose info. source ensemble is

$$\left\{ \left(\tfrac{1}{2}, |0\rangle\right), \left(\tfrac{1}{2}, |+\rangle\right) \right\}$$

Shannon compression would require 1 bit/symbol for compression.

Schumacher compression requires significantly less.

density operator is

$$\tfrac{1}{2}|0\rangle\langle 0| + \tfrac{1}{2}|+\rangle\langle +| \quad \text{which}$$

diagonalizes as

$$\cos^2(\pi/8)\,|+'\rangle\langle +'| + \sin^2(\pi/8)\,|-'\rangle\langle -'|$$

where $|+'\rangle = \cos(\pi/8)|0\rangle + \sin(\pi/8)|1\rangle$

& $|-'\rangle = \sin(\pi/8)|0\rangle - \cos(\pi/8)|1\rangle$

can compress down to

$$H_2(\cos^2(\pi/8)) \approx 0.6 \text{ qubits/symbol}$$

What if source instead issues

$$\left\{ \left(\tfrac{1}{2}, |0\rangle\langle 0| \otimes |0\rangle\langle 0|\right), \left(\tfrac{1}{2}, |1\rangle\langle 1| \otimes |+\rangle\langle +|\right) \right\}$$

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Entanglement Concentration

- Given that entanglement is useful, what if Alice & Bob share pure states that are not maximally entangled?

- If they have many copies, they can "concentrate" them to ebits at a rate given by "entropy of entanglement" (von Neumann entropy of half of a bipartite state)

- interprets the term "ebit" in an information theoretic way

- Begin w/ a simple example.

  Suppose Alice & Bob share

  $$|\phi\rangle^{AB} = \cos\theta\, |00\rangle^{AB} + \sin\theta\, |11\rangle^{AB} \quad (\text{where } \theta \neq \pi/4)$$

  (this is most general form from Schmidt decomposition)

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Consider taking 3 copies of this state.
state naturally breaks down into subspaces:

$$|\phi\rangle^{A_1B_1}|\phi\rangle^{A_2B_2}|\phi\rangle^{A_3B_3} =$$

$$\left(\cos\theta\,|00\rangle + \sin\theta\,|11\rangle\right)^{\otimes 3}$$

$$\begin{aligned}
= \; & \cos^3\theta \; |000\rangle^A |000\rangle^B + \\
& \cos\theta \sin^2\theta \left(|110\rangle^A |110\rangle^B + |101\rangle^A |101\rangle^B + |011\rangle^A |011\rangle^B \right) + \\
& \cos^2\theta \sin\theta \left(|100\rangle^A |100\rangle^B + |010\rangle^A |010\rangle^B + |001\rangle^A |001\rangle^B \right) + \\
& \sin^3\theta \; |111\rangle^A |111\rangle^B
\end{aligned}$$

rewrite one more time:

$$\begin{aligned}
= \; & \cos^3\theta \; |000\rangle^A |000\rangle^B + \\
& \sqrt{3}\cos\theta\sin^2\theta \left[\frac{1}{\sqrt{3}} \left(|110\rangle^A |110\rangle^B + |101\rangle^A |101\rangle^B + |011\rangle^A |011\rangle^B \right) \right] \\
& \sqrt{3}\cos^2\theta\sin\theta \left[\frac{1}{\sqrt{3}} \left(|100\rangle^A |100\rangle^B + |010\rangle^A |010\rangle^B + |001\rangle^A |001\rangle^B \right) \right] + \\
& \sin^3\theta \; |111\rangle^A |111\rangle^B
\end{aligned}$$

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- can naturally divide Hilbert space into 4 subspaces, by Hamming weight of Alice or Bob's state.
- projectors are

$$|000\rangle\langle 000|^A,$$

$$|001\rangle\langle 001|^A + |010\rangle\langle 010|^A + |100\rangle\langle 100|^A,$$

$$|110\rangle\langle 110|^A + |101\rangle\langle 101|^A + |011\rangle\langle 011|^A,$$

$$|111\rangle\langle 111|^A$$

after projecting w/ this measurement, they get

$$|000\rangle|000\rangle \quad \text{or}$$

$$\frac{1}{\sqrt{3}}\left\{|001\rangle|001\rangle + |100\rangle|100\rangle + |010\rangle|010\rangle\right\} \quad \text{or}$$

$$\frac{1}{\sqrt{3}}\left\{|110\rangle|110\rangle + |101\rangle|101\rangle + |011\rangle|011\rangle\right\} \quad \text{or}$$

$$|111\rangle|111\rangle$$

these are maximally entangled states

More generally, we might have n copies, for which we can write the state as

$$|\phi\rangle^{A^n B^n} = \sum_{k=0}^{n} \sqrt{\binom{n}{k}} \cos^{n-k}(\theta) \sin^k\theta \cdot \left[\frac{1}{\sqrt{\binom{n}{k}}} \sum_{x: w(x)=k} |x\rangle^{A^n} |x\rangle^{B^n}\right] \quad (*)$$

where $w(x)$ is Hamming weight.

How does protocol work for large n?

- need a handle on $\binom{n}{k}$ when $\underline{n}$ is large. Use Stirling's approx.

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

$$\therefore \frac{n!}{n-k!\,k!} \sim \frac{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n}{\sqrt{2\pi(n-k)}\left(\frac{n-k}{e}\right)^{n-k}\sqrt{2\pi k}\left(\frac{k}{e}\right)^k}$$

$$= \sqrt{\frac{n}{2\pi(n-k)k}}\ \frac{n^n}{(n-k)^{n-k}k^k}$$

$$= \sqrt{\frac{n}{2\pi(n-k)k}}\left(\frac{n-k}{n}\right)^{-(n-k)}\left(\frac{k}{n}\right)^{-k}$$

$$= \sqrt{\frac{n}{2\pi(n-k)k}}\, 2^{n\left[\frac{-(n-k)}{n}\log\frac{n-k}{n} - \frac{k}{n}\log\frac{k}{n}\right]}$$

$$= \sqrt{\frac{n}{2\pi(n-k)k}}\, 2^{n H_2\left(\frac{k}{n}\right)}$$

when n is large, this term dominates polynomial factor on Left.

- we can now develop a strategy.
  Alice first performs typical subspace measurement on her share (Bob can do the same, but it won't change anything).
- measurement is for strong typicality, succeeds w/ high probability (LLN)

  state collapses to

$$\sum_{\substack{k=0:\\ \left|\frac{k}{n}-\sin^2\theta\right|<\delta,\\ \left|\frac{n-k}{n}-\cos^2\theta\right|<\delta}}^{n} \quad (*)$$

- Alice then measures Hamming weight k
- Bob does same step & gets same result

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State collapses to

$$\frac{1}{\sqrt{\text{poly}(n)\, 2^{n H_2(k/n)}}} \sum_{x\,:\,w(x)=k} |x\rangle^{A^n} |x\rangle^{B^n}$$

this is maximally entangled qudit w/ Schmidt rank

$$2^{n H_2\left(\frac{k}{n}\right)} \geq 2^{n\left[H_2(\cos^2\theta) - \delta'\right]}$$

from Fannes' inequality

(poly(n) factor doesn't matter asymptotically)

this gives approximately

$n H_2(\cos^2\theta)$ ebits

Argument for qudits is similar.

1) take Schmidt decomposition & many copies

$$\left(|\phi\rangle^{AB}\right)^{\otimes n} = \left(\sum_x \sqrt{p(x)}\, |x\rangle^A |x\rangle^B\right)^{\otimes n} = \sum_{x^n} \sqrt{p(x^n)}\, |x^n\rangle^A |x^n\rangle$$

2) do typical subspace measurement

3) do measurement of empirical distribution

4) state collapses to maximally entangled qudit of Schmidt rank $\approx 2^{n H(A)_\phi}$

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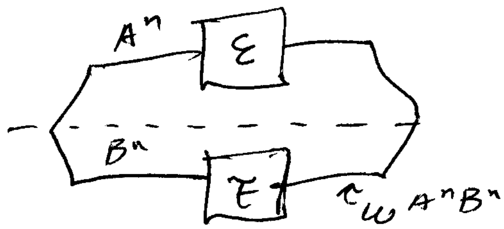
Converse Proof

Most general protocol:



At end, state $\omega^{A^nB^n}$ should be close to $\Phi^{\hat{A}\hat{B}}$:

$$\|\omega^{A^nB^n} - \Phi^{\hat{A}\hat{B}}\|_1 \leq \epsilon$$

$$\begin{aligned}
2nE &= 2H(\hat{A})_\Phi \\
&= H(\hat{A})_\Phi + H(\hat{B})_\Phi - H(\hat{A}\hat{B})_\Phi \\
&= I(\hat{A};\hat{B})_\Phi \\
&\leq I(\hat{A};\hat{B})_\omega + n\epsilon' \quad \text{(Fannes')} \\
&\leq I(A^n;B^n)_{\varphi^{\otimes n}} + n\epsilon' \quad \text{(QDP)} \\
&= H(A^n)_{\varphi^{\otimes n}} + H(B^n)_{\varphi^{\otimes n}} \\
&\quad - H(A^nB^n)_{\varphi^{\otimes n}} + n\epsilon' \\
&= 2nH(A)_\varphi
\end{aligned}$$

$\square$